Toroidal orbifold resolutions and intersection numbers with Gauged Linear Sigma Models

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Abstract

We resolve $T^6/\mathbb{Z}_3$ orbifold singularities with Gauged Linear Sigma Models (GLSMs) and describe the factorizable underlying six-torus $T^2 \times T^2 \times T^2$ as the intersection set of three elliptic curves in a projective space. Divisors are given as hypersurface equations that depend on the chosen resolution model. We determine intersection numbers in an algebraic way counting the solutions to the equation system formed by the elliptic curves defining the six-torus and the hypersurface constraints of the corresponding divisors. This procedure is extended to various resolution phases in the so-called minimal fully resolvable model.

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Contents

Chapter 1

Introduction

To date, there are four known fundamental interactions in physics that lay down the laws of every physical process we can observe. Namely, these interactions are Electromagnetism, Weak and Strong interactions and Gravity. Particle physics can be described very accurately by a quantum theory of fields which is called the Standard Model (SM). This theory arises from some Lagrangian possessing a local gauge invariance under the gauge group $G_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y$. Each of these gauge group factors gives rise to so-called gauge bosons which are spin 1 particles mediating a corresponding fundamental interaction: The QCD gauge group $SU(3)_C$ creates the eight gluons which carry the strong force gluing together the quarks in every atomic nucleus. The last two factors $SU(2)_L \times U(1)_Y$ generate the massive $W^\pm$ and Z^0 bosons and the photon γ, which describe the weak interaction and quantum electrodynamics, the quantum theory of electromagnetism, in a unified framework called electroweak theory.

Gravitational interactions ar best described within Einsteins theory of General Relativity (GR). It explains gravity in a geometrical way assuming our four-dimensional spacetime to be curved. The Standard Model and General Relativity, considered both seperately on very small and very big length scales respectively, can describe almost every observable physical process by an accuracy that is higher than what we can experimentally confirm.

At high energies, however, there have to be physical processes that both involve gravitational and quantum field theoretical effects at the same time, so we have to find a quantized theory of gravity. But, with our present knowledge, this is impossible: a quantum field theory of gravity would require an infinite number of counter terms and is thus a non-predictive theory.

The aim of string theory is nothing less than finding a unified theory of gravity and the other three fundamental interactions we know. It is probable that such a theory will explain all physical processes in our universe based on a very few number of axioms and free parameters. Thus, in popular science, it is often referred to as the "theory of everything" or "final theory", meaning that it has the potential to explain the laws of nature how they really are, and not just to give a very accurate description of the processes we see.

The basic concept of string theory is that elementary particles do not correspond to point-like, but to tiny one-dimensional objects that can carry various vibration modes, comparable to the vibrating string of a violin. A crucial point is that such a theory is only consistent in 10 or 26 spacetime dimensions, whereas the latter case comes along with tachyons which are particles that move faster than the speed of light. Thus, it is more likely that string models in 10 spacetime dimensions are potential descriptions of the physics in our universe.

An open question is how to "hide" the six extra dimensions so that we can not observe them in our everyday life. We consider the 10-dimesional spacetime $\mathcal{M}_{10}$ to be factorized into the two parts $\mathcal{M}_{10} = \mathcal{M}_4 \otimes \mathcal{M}_6$, where $\mathcal{M}_4$ represents the ordinary four-dimensional Minkowskian spacetime and $\mathcal{M}_6$ corresponds to the space of the further six dimensions the theory must have. A very direct way to hide these extra dimensions is to compactify them, that means to give them a finite and small enough volume. The easiest compact space one can imagine is a flat torus, where we simply endow all the coordinates with periodical identifications.

Another possibility is to consider $\mathcal{M}_6$ to be a so-called orbifold, which we will explain in the subsequent section. Orbifold compactifications lead to a much more realistic phenomenology concerning spacetime and gauge symmetry breaking [3] than a torus compactification does. As we will see soon, each orbifold possesses so-called orbifold fixed points. Since it is impossible to define geometrical quantities like a metric on these points, they form singularities that have to be resolved for a proper description of string propagation in this space.

The resolution of such orbifold singularities form one main part of this work. A Gauged Linear Sigma Model naturally leads to constraints (the so-called D-term constraints) that exclude orbifold singularities from the target space for some specific values of free FI-parameters present in this model. Since the resolution depends on the value of these free parameters, one can investigate the resolution procedure in a continuous manner. In both the resolved and the non-resolved orbifold one can define real codimension 2 hypersurfaces which are called divisors. Another main part is to determine how often these divisors intersect with each other, i.e. how many points lie in the intersection set of $\frac{dim(\mathcal{M}_6)}{2} = 3$ such divisors. We will see that intersection numbers can change going from one resolution phase to another, i.e. varying the free FI-parameters from one specific regime to another.

In chapter 2, we discuss some basic properties of orbifolds, in particular of $T^2/\mathbb{Z}_3$ and $T^6/\mathbb{Z}_3$. Furthermore, we present the basics of Gauged Linear Sigma Models (GLSMs) which can be used to resolve orbifold singularities. In chapter 3, we describe another orbifold resolution procedure which is based on toric geometry as it was done in [9]. It is also possible to compute intersection numbers using this resolution method. Finally, in chapter 4, we use GLSM methods to resolve the $T^6/\mathbb{Z}_3$ orbifold in different so-called resolution models and compute intersection numbers in resolved and non-resolved phases. For example, we give the intersection numbers of various phases of the minimal fully resolvable model and even compute an intersection number of a $T^6/\mathbb{Z}_3$ orbifold on a non-factorized E_6 torus lattice.

Chapter 2

Background material

2.1 The $T^2/\mathbb{Z}_3$ and $T^6/\mathbb{Z}_3$ orbifolds

An orbifold is a generalized manifold. One of the main properties of a manifold of dimension n is that it locally looks like an open subset of $\mathbb{R}^n$. Without giving any rigorous mathematical definition, an orbifold can be seen as a manifold that can not in general locally be described by an open subset of $\mathbb{R}^n$ but by a quotient by a finite group thereof.

Let us consider the $T^2/\mathbb{Z}_3$ orbifold. To describe this orbifold we introduce the coordinate $u \in \mathbb{C}$. In order to obtain a two-torus T^2, we have to make the periodic identifications

$$u \sim u + 1, \qquad u \sim u + \tau, \tag{2.1}$$

with some complex structure τ, $Im(\tau) \neq 0$.

Next, we divide out the $\mathbb{Z}_3$-orbifold action θ. This means we have to perform a further identification, namely

$$\theta : u \sim \zeta u, \qquad \zeta = e^{\frac{2\pi i}{3}}. \tag{2.2}$$

This further symmetry constrains the possible values of the complex structure τ of the underlying two-torus T^2. The torus lattice given by

$$\Lambda = \{m + n\tau \mid m, n \in \mathbb{Z}\} \tag{2.3}$$

has to be invariant under the orbifold action θ. It is not hard to show that $\tau = \zeta$ is a possible choice for the complex strucutre τ. This is all we need to define $T^2/\mathbb{Z}_3$. We see that there are points that are invariant under the orbifold action: if we take for example the point $u = f_1 = 0$, it is obvious that it does not change under the action (2.2). Furthermore, the points $u = f_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}i = \frac{\zeta+2}{3}$ and $u = f_3 = \frac{\sqrt{3}}{3}i = 2\frac{\zeta+2}{3}$ are invariant under combinations of the orbifold action (2.2) and the torus symmetries (2.1). These so-called orbifold fixed points are plotted as red spots in figure

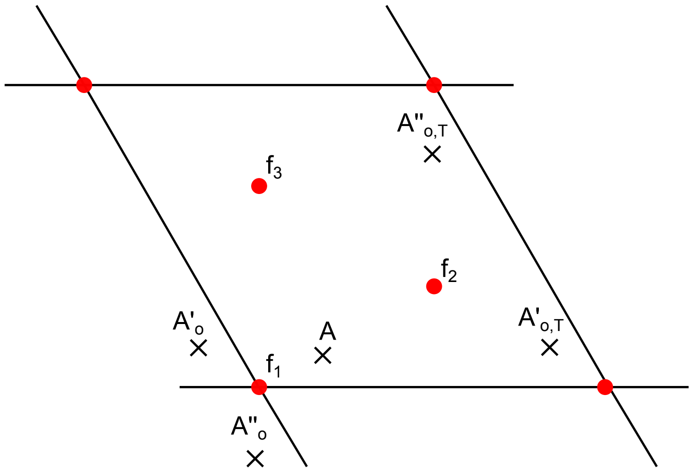

Figure 2.1: The $T^2/\mathbb{Z}_3$ orbifold. The red points are the orbifold fixed points. As an example, we marked the point A and its images under the orbifold action, A'_o and A''_o. Using the torus identifications (2.1), we can map them back to the torus we started and obtain the points $A'_{o,T}$ and $A''_{o,T}$.

2.1. Since it is impossible to define geometrical quantities like a metric at such orbifold fixed points, they correspond to orbifold singularities. It is the aim of every orbifold resolution process to smooth out these singularities. Further information on orbifolds can be found in the literature, for example in [4], [5].

To have a coordinate description of $T^2/\mathbb{Z}_3$ where the torus symmetries (2.1) are already built-in (i.e. where points that are identified by (2.1) have the same coordinates), we map our coordinate u to an elliptic curve in a projective space $\mathbb{P}^2_{1,1,1}$ in the next section.

2.1.1 The Weierstraß mapping

Two-dimensional tori can nicely be described by elliptic curves in a weighted projective space $\mathbb{P}^2_{p,q,1}$. A two-torus with the periodic identifications $u \sim u+1 \sim u+\tau$ is mapped in such a weighted projective space using the Weierstraß function $\wp_\tau(u)$. We review just some of the main properties of this function here, further details can be found in textbooks, for example in [6].

The Weierstraß function $\wp_\tau(u)$ is a double periodic complex function $\wp_\tau(u): \mathbb{C} \to \mathbb{C}$ with the periocities $\wp_\tau(u) = \wp_\tau(u+1) = \wp_\tau(u+\tau)$ for some complex structure τ. An explicit form is given by the expansion

$$\wp_\tau(u) = \frac{1}{u^2} + \sum_{(m,n)\neq(0,0)} \left\{ \frac{1}{(u+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right\}. \tag{2.4}$$

We see that this is simply the function $1/u^2$ which becomes invariant under the lattice $\Lambda = \{\lambda = m + n\tau \mid m, n \in \mathbb{Z}\}$ by adding each term of $1/\hat{u}^2$, where $u = \hat{u} \mod \lambda,\ \lambda \in \Lambda$. The terms $-\frac{1}{(m+n\tau)^2}$ are needed for the series to be convergent.

The first derivative of the Weierstraß $\wp$-function is

$$\wp'_\tau(u) = -2\sum_{m,n} \frac{1}{(u+m+n\tau)^3}. \tag{2.5}$$

This function solves the Weierstraß differential equation

$$[\wp'_\tau]^2(u) = 4\wp^3_\tau(u) + f(\tau)\wp_\tau(u) - g(\tau), \tag{2.6}$$

where the functions $f(\tau)$ and $g(\tau)$ are

$$f(\tau) = 4(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3), \qquad g(\tau) = 4\epsilon_1\epsilon_2\epsilon_3. \tag{2.7}$$

We call $\epsilon_1 = \wp_\tau(1/2), \epsilon_2 = \wp_\tau(\frac{\tau}{2})$ and $\epsilon_3 = \wp_\tau(\frac{1+\tau}{2})$.

A weighted projective space $\mathbb{P}^2_{p,q,1}$ has three homogeneous coordinates (x, y, v). Hence at first sight this looks as we intend to perform a mapping with one complex degree of freedom u in the domain to three (x, y, v) in the image space. But since we map to a weighted projective space, we have a $\mathbb{C}^*$-symmetry acting on the coordinates as

$$\mathbb{C}^* : \qquad (x, y, v) \sim (\lambda^p x, \lambda^q y, \lambda v), \qquad \lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}, \tag{2.8}$$

reducing the number of degrees of freedom in the image space from 3 to 2. To construct an isomorphism from our two-torus to an elliptic curve in a weighted projective space, there can not be a dimensional mismatch. Therefore, we perform a mapping using $\wp_\tau(u)$ and its derivative $\wp'_\tau(u)$ whereby we can use (2.6) as a further constraint reducing our image space dimension from 2 to 1, which makes an isomorphism possible.

The concrete case: $T^2/\mathbb{Z}_3$

As mentioned above, in the $T^2/\mathbb{Z}_3$ case, a possible choice for the complex structure is $\tau = \zeta = e^{2\pi i/3}$. Furthermore, it turns out that the function $f(\tau)$ in (2.7) is zero. The mapping from $T^2_{\tau=\zeta} \to \mathbb{P}^2_{1,1,1}$ can be performed by [7]

$$u \to (x, y, v) = \begin{cases} \left(\wp_\zeta(u), \epsilon_1^{-1/2}\wp'_\zeta(u)/2, \epsilon_1\right), & \text{away from lattice points,} \\ (0, 1, 0) & \text{on the lattice, } u \in \Lambda, \\ \left(\frac{2\epsilon_1^{1/2}\wp_\zeta(u)}{\wp'_\zeta(u)}, 1, \frac{2\epsilon_1^{3/2}}{\wp'_\zeta(u)}\right) & \text{close to lattice points.} \end{cases} \tag{2.9}$$

The Weierstraß differential equation (2.6) then takes the form

$$y^2 v = x^3 - v^3. \tag{2.10}$$

By a linear change of coordinates this can be brought to a form where the $\mathbb{Z}_3$-symmetries are manifest. This is done by [7]

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 1 \\ 0 & -\frac{1}{\sqrt{3}} & 1 \end{pmatrix} \begin{pmatrix} x \\ 2^{-1/3} y \\ 2^{-1/3} v \end{pmatrix}, \tag{2.11}$$

which brings (2.10) to the form

$$z_1^3 + z_2^3 + z_3^3 = 0. \tag{2.12}$$

We will mostly use these coordinates for the rest of this work.

In this description we immediately see three independent $\mathbb{Z}_3$ actions, namely $z_i \to \zeta z_i, i = 1, 2, 3$. However, we choose our basis of $\mathbb{Z}_3$-actions to be

$$\begin{aligned} \mathbb{C}^* : &\qquad (z_1, z_2, z_3) \to (\zeta z_1, \zeta z_2, \zeta z_3), \\ \theta : &\qquad (z_1, z_2, z_3) \to (\zeta z_1, z_2, z_3), \\ \alpha : &\qquad (z_1, z_2, z_3) \to (z_1, \zeta^2 z_2, \zeta z_3). \end{aligned} \tag{2.13}$$

The first symmetry is already contained in the $\mathbb{C}^*$ symmetry of the weighted projective space $\mathbb{P}^2_{1,1,1}$ we are in, so this action does not carry any further information. Using the relations $\wp_\zeta(\zeta u) = \zeta \wp_\zeta(u)$ and $\wp'_\zeta(\zeta u) = \wp'_\zeta(u)$, it is not hard to identify the second one as the orbifold action $\theta : u \to \zeta u$, simply by mapping ζu into our new coordinates. The third action is a bit more subtle but has been identified in Appendix A.2 of [7] to be the so-called 3-volution α that acts as $u \to u + \frac{\zeta - 1}{3}$.

We see that, if we mod out the $\mathbb{Z}_3$ orbifold action θ only, the three orbifold fixed points $u_i = i \cdot \frac{\zeta+2}{3}$, $i = 0, 1, 2$ are obtained in Weierstraß coordinates by setting z_1 to zero, because points fulfilling that condition are invariant under the action θ given in (2.13). To see that we have indeed 3 orbifold fixed points, we compute the solutions to the $\mathbb{C}^*$ invariant term $\left(\frac{z_2}{z_3}\right)^3 = -1$. They are given by $\frac{z_2}{z_3} = -\zeta^n$, $n = 0, 1, 2$. Hence we exactly get 3 orbifold fixed points.

Contrary, if we mod out the actions θ and α together, we have all the three actions of (2.13) which form a basis of $\mathbb{Z}_3$-actions on the coordinates z_i. We are free to change this basis of $\mathbb{Z}_3$-actions back to

$$\theta_i : \qquad z_i \to \zeta z_i, i = 1, 2, 3, \tag{2.14}$$

where it is obvious that the three orbifold fixed points (the fixed points under these actions) are given by $z_i = 0$, $i = 1, 2$ or 3.

We can also mod out the 3-volution action α only. Since the action α changes the phase of the two coordinates z_2, z_3 simultaneously, we have to set them both to zero to get the fixed points under this action, $z_2 = z_3 = 0$. Because of (2.12), this directly entails that also $z_1 = 0$. But the point $z_1 = z_2 = z_3 = 0$ is excluded from the projective space $\mathbb{P}^2_{1,1,1}$, so we have produced a contradiction. This means that there are no fixed points under the 3-volution α only, which we already knew because α acts free on the coordinates u as the shift $u \to u + \frac{\zeta-1}{3}$.

Generalization to $T^6/\mathbb{Z}_3$

The generalization of this choice of coordinates to a six-dimensional torus $T^6 = T^2 \times T^2 \times T^2$ is easy: instead of one complex coordinate u describing our torus, we now have three of them, all satisfying the equivalence relations $u_a \sim u_a + 1 \sim u_a + \tau, \quad a = 1,2,3$. The orbifold action θ then can be introduced as

$$\theta: \qquad (u_1, u_2, u_3) \to (\zeta u_1, \zeta u_2, \zeta u_3) \tag{2.15}$$

and the 3-volutions α_a (now there are three 3-volutions possible, each one acting independently on one of the coordinates u_a) are

$$\alpha_a: \qquad u_a \to u_a + \frac{\zeta - 1}{3}. \tag{2.16}$$

We simply map every coordinate u_a as before to obtain the set of equations

$$z_{a1}^3 + z_{a2}^3 + z_{a3}^3 = 0, \quad a = 1,2,3, \tag{2.17}$$

describing our T^6 with complex structure $\tau = \zeta$. Analogously to the two-dimensional case, the orbifold action θ takes the form

$$\theta: \qquad (z_{11}, z_{21}, z_{31}) \to (\zeta z_{11}, \zeta z_{21}, \zeta z_{31}), \tag{2.18}$$

and the 3-volutions look like

$$\alpha_a: \qquad (z_{a1}, z_{a2}, z_{a3}) \to (z_{a1}, \zeta^2 z_{a2}, \zeta z_{a3}), \tag{2.19}$$

where it is important to note that there is only one $\mathbb{Z}_3$ orbifold action θ acting on all the three two-tori simultaneously. So we indeed get a $T^6/\mathbb{Z}_3$ orbifold. We must not confuse $T^6/\mathbb{Z}_3$ with $\left(T^2/\mathbb{Z}_3\right)^3$, where we would have three $\mathbb{Z}_3$ orbifold actions θ_i acting independently on the three two-tori as

$$\theta_i: \qquad z_i \to \zeta z_i, \qquad i = 1,2 \text{ or } 3.$$

An orbifold fixed point must again fulfill the conditions we already had for $T^2/\mathbb{Z}_3$: $z_{a1} = 0$, $\frac{z_{a2}}{z_{a3}} = -\zeta^{n_a}$ $\forall a = 1,2,3$ if we divide out the orbifold action θ only and $z_{an_a} = 0$, $\forall a = 1,2,3$ when we divide out both the actions θ and α_a. So in each two-torus we have three possibilities, $n_a = 1,2$ or 3. Thus, using simple combinatorics, we see that $T^6/\mathbb{Z}_3$ has $3 \cdot 3 \cdot 3 = 27$ fixed points.

2.2 Orbifold resolutions with Gauged Linear Sigma Models

We consider a gauged linear sigma model with $(2,2)$ supersymmetry where the Lagrangian can be written as the sum [8]

$$L = L_{\text{kin}} + L_W + L_{\text{gauge}} + L_{D,\theta}, \tag{2.20}$$

with L_{kin} the kinetic energy of the chiral superfields, L_W the superpotential interaction, L_{gauge} the kinetic energy of the gauge fields and $L_{D,\theta}$ a Fayet-Iliopoulos term and θ-angle.

The kinetic term L_{kin} has the form

$$L_{\text{kin}} = \int d^4\theta \bar{\mathcal{Z}}\mathcal{Z}, \tag{2.21}$$

with the chiral superfield $\mathcal{Z}$,

$$\mathcal{Z} = z + \theta^\alpha \Psi_\alpha + \theta^\alpha \theta_\alpha F. \tag{2.22}$$

To obtain a local $U(1)$ gauge invariant form under the gauge transformation $\mathcal{Z} \to e^{iA}\mathcal{Z}$ with the chiral superfield $A = A(x^\mu, \theta^\pm, \bar{\theta}^\pm)$ [12], we have to add a factor e^V in between $\bar{\mathcal{Z}}$ and $\mathcal{Z}$,

$$L_{\text{kin}} = \int d^4\theta \bar{\mathcal{Z}} e^V \mathcal{Z}. \tag{2.23}$$

In order to get the local $U(1)$ gauge symmetry, the field V has to be a real superfield $V(x^\mu, \theta^\pm, \bar{\theta}^\pm)$ that transforms as

$$V \to V + i(\bar{A} - A). \tag{2.24}$$

In Wess-Zumino gauge, such a field V can be expanded in the superspace coordinates θ [12] as

$$\begin{aligned} V = {} & \theta^-\bar{\theta}^-(v_0 - v_1) + \theta^+\bar{\theta}^+(v_0 + v_1) - \theta^-\bar{\theta}^+\sigma - \theta^+\bar{\theta}^-\bar{\sigma} + \\ & + i\theta^-\theta^+(\bar{\theta}^-\bar{\lambda}_- + \bar{\theta}^+\bar{\lambda}_+) + i\bar{\theta}^+\bar{\theta}^-(\theta^-\lambda_- + \theta^+\lambda_+) + \\ & + \theta^-\theta^+\bar{\theta}^+\bar{\theta}^- D, \end{aligned} \tag{2.25}$$

where v_0, v_1 are vector fields, σ is a complex scalar field, $\lambda_\pm$, $\bar{\lambda}_\pm$ are fermionic fields and D is a real scalar field.

So the superspace integral for L_{kin} in (2.23) can be evaluated to [12]

$$\begin{aligned} L_{\text{kin}} = {} & \int d^4\theta \bar{\mathcal{Z}} e^V \mathcal{Z} = -D^\mu \bar{z} D_\mu z + i\bar{\Psi}_-(D_0 + D_1)\Psi_- + \\ & + i\bar{\Psi}_+(D_0 - D_1)\Psi_+ + D|z|^2 + |F|^2 - |\sigma|^2|z|^2 - \bar{\Psi}_-\sigma\Psi_+ + \\ & - \bar{\Psi}_+\bar{\sigma}\Psi_- - i\bar{z}\lambda_-\Psi_+ + i\bar{z}\lambda_+\Psi_- + i\bar{\Psi}_+\bar{\lambda}_- z - i\bar{\Psi}_-\bar{\lambda}_+ z, \end{aligned} \tag{2.26}$$

where D_μ denotes the covariant derivative, $D_\mu := \partial_\mu + iv_\mu$.

Analogously, the components L_{gauge} and $L_{D,\theta}$ can be written as [12]

$$\begin{aligned} L_{\text{gauge}} &= \frac{1}{2e^2}(-\partial^\mu\bar{\sigma}\partial_\mu\sigma + i\bar{\lambda}_-(\partial_0+\partial_1)\lambda_- + \\ &\quad + i\bar{\lambda}_+(\partial_0-\partial_1)\lambda_+ + (\partial_0 v_1 - \partial_1 v_0)^2 + D^2), \end{aligned} \tag{2.27}$$

$$L_{D,\theta} = -bD + \theta(\partial_0 v_1 - \partial_1 v_0), \tag{2.28}$$

where e^2 is the coupling constant which has mass dimension 1. b is a Fayet-Iliopoulos parameter and θ the theta-angle, which must not be confused with any superspace coordinate.

Making use of the equations of motion, we can get rid of the fields D and F in $L = L_{\text{kin}} + L_{\text{gauge}} + L_{D,\theta}$ [12]. We get

$$\begin{aligned} L = &-D^\mu\bar{z}D_\mu z + i\bar{\Psi}_-(D_0+D_1)\Psi_- + i\bar{\Psi}_+(D_0-D_1)\Psi_+ + \\ &- \frac{e^2}{2}(|z|^2-b)^2 - |\sigma|^2|z|^2 - \bar{\Psi}_-\sigma\Psi_+ - \bar{\Psi}_+\bar{\sigma}\Psi_- + \\ &- i\bar{z}\lambda_-\Psi_+ + i\bar{z}\lambda_+\Psi_- + i\bar{\Psi}_+\bar{\lambda}_- z - i\bar{\Psi}_-\bar{\lambda}_+ z + \\ &+ \frac{1}{2e^2}(-\partial^\mu\bar{\sigma}\partial_\mu\sigma + i\bar{\lambda}_-(\partial_0+\partial_1)\lambda_- + \\ &+ i\bar{\lambda}_+(\partial_0-\partial_1)\lambda_+ + (\partial_0 v_1 - \partial_1 v_0)^2) + \theta(\partial_0 v_1 - \partial_1 v_0), \end{aligned} \tag{2.29}$$

where we identify the potential energy of the scalar fields σ and z

$$U = \frac{e^2}{2}(|z|^2-b)^2 + |\sigma|^2|z|^2. \tag{2.30}$$

Generalizing that to the case of multiple charged chiral superfields $\mathcal{Z}_a$ and considering a nonvanishing interaction potential contribution

$$L_W = \int d^2\theta W(\mathcal{Z}_a) + c.c. = \frac{|m|^2}{2}\left(\left|\frac{\partial W}{\partial z_i}\right|^2 + \frac{\partial^2 W}{\partial z_i \partial z_j}\Psi_{-,i}\Psi_{+,j}\right) + c.c., \tag{2.31}$$

we get [12]

$$U \sim \frac{e^2}{2}\left(\sum_a q_a\,|z_a|^2 - b\right)^2 + |m|^2\sum_a |W_a|^2, \tag{2.32}$$

where the charge q_a of a chiral superfield $\mathcal{Z}_a$ means that it transforms as $\mathcal{Z}_a \to e^{iq_a A}\mathcal{Z}_a$ and $W_a = \frac{\partial W}{\partial z_a}$.

Since string theory is a scale invariant theory but e and m are dimensionful quantities, we have to take the conformal limit $e, m \to \infty$. This on the other hand leads to the constraints

$$\sum_a q_a\,|z_a|^2 - b = 0 \tag{2.33}$$

Superfield	$\mathcal{Z}_1$	$\mathcal{Z}_2$	$\mathcal{Z}_3$	$\mathcal{X}$
U(1) charge q	1	1	1	-3

Table 2.1: Charge assignment for the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold GLSM resolution

and

$$W_a = \frac{\partial W}{\partial z_a} = 0. \tag{2.34}$$

These are the so-called D- and F-term constraints, respectively.

It is a crucial point that the complex scalar components of the chiral superfields are interpreted as target space coordinates. This is why we can use the superpotential W to reproduce the torus-defining equations (2.17) in the F-term constraints.

2.2.1 GLSM resolution of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold

To see how the GLSM resolution process works, we consider the noncompact $\mathbb{C}^3/\mathbb{Z}_3$ orbifold as a simple example. Since all the 27 orbifold singularities of $T^6/\mathbb{Z}_3$, which we will investigate lateron, locally look like $\mathbb{C}^3/\mathbb{Z}_3$, it is a very appropriate example to begin with.

The $\mathbb{C}^3/\mathbb{Z}_3$ orbifold is described by $(z_1, z_2, z_3) \in \mathbb{C}^3$ divided by a discrete $\mathbb{Z}_3$ orbifold action θ,

$$\theta : \qquad (z_1, z_2, z_3) \sim (\zeta z_1, \zeta z_2, \zeta z_3). \tag{2.35}$$

We introduce the chiral superfields $\mathcal{Z}_i$, $i = 1, 2, 3$ and interpret their scalar components as target space coordinates. To implement the orbifold symmetry, we have to endow the fields $\mathcal{Z}_i$ with equal charges m. Since this would modify the target space dimension, we have to add a new chiral superfield $\mathcal{X}$ with the charge $-\sum m$ to our resolution process. Furthermore, this allows us to construct the orbifold symmetries we want to have. The charge assignment is given in table 2.1. From the charge assignment we can directly read off the D-term constraint

$$D : \qquad |z_1|^2 + |z_2|^2 + |z_3|^2 - 3|x|^2 = b. \tag{2.36}$$

Now everything depends on the value of the FI-parameter b.

$b < 0$: The orbifold phase

For $b < 0$, the exceptional coordinate x (the scalar component of the added chiral superfield $\mathcal{X}$) has to have a vacuum expectation value (VEV), hence we are able to gauge fix its $U(1)$ phase making use of the gauge freedom

$$x \sim e^{-3\cdot 2\pi i\phi}x, \qquad z_i \sim e^{2\pi i\phi}z_i, \qquad i = 1, 2, 3. \tag{2.37}$$

Suppose we gauge fix the phase of x to some definite value. We are free to write this as

$$\left\{x \sim e^{-3\cdot 2\pi i\phi}x \quad |e^{-3\cdot 2\pi i\phi} \overset{!}{=} 1\right\}. \tag{2.38}$$

This means that $\phi = \frac{n}{3}$, $n \in \mathbb{Z}$. So there is still a $\mathbb{Z}_3$ gauge-freedom left over on the z_i coordinates, namely

$$z_i \sim e^{2\pi i \frac{n}{3}} z_i \qquad i = 1, 2, 3. \tag{2.39}$$

This is nothing other than the orbifold action (2.35), hence this phase is called the orbifold phase.

$b > 0$: The blow-up phase

In this phase, because of the D-term constraint (2.36), at least one z_i has to be nonzero. Thus the orbifold singularity $(z_1, z_2, z_3) = (0, 0, 0)$ is not existent anymore in the blow-up phase, the orbifold singularity has been resolved. To be precise, if we assume that $x = 0$, it has been blown-up to a 5-sphere S^5 whose size is controlled by the parameter b,

$$b = |z_1|^2 + |z_2|^2 + |z_3|^2. \tag{2.40}$$

All points within this sphere are excluded from the target space, whereas all the outlying points are accessible. We can easily see this looking at (2.36) again:

$$b + 3|x|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2.$$

This equation allows for points lying on the sphere with radius b described in (2.40) for $x = 0$, but also for points that lie on a sphere with radius $b' = b + 3|x|^2 > b$ for $x \neq 0$. Hence all points at the outside of the sphere in (2.40) are allowed because x is a free coordinate without any upper limit.

In the limit $b \searrow 0$ we get the $\mathbb{C}^3/\mathbb{Z}_3$-orbifold with the orbifold fixed point excluded, $\left(\mathbb{C}^3/\mathbb{Z}_3\right) \setminus \{0\}$.

2.2.2 Divisors

A divisor is a complex codimension 1 hypersurface living in an orbifold. In [9], three different types of divisors were specified. The ordinary divisors, denoted D_{ai}, correspond to hypersurfaces where we fix one complex coordinate to the value of an orbifold fixed point. The inherited divisors R_a also fix one of the three orbifold coordinates to a definite value, but this time away from any orbifold fixed point. Furthermore, there are the so-called exceptional divisors E, which arise in the resolution process of the orbifold singularities. In the GLSM language, the exceptional divisors E correspond to zeros of the exceptional coordinates x, $x = 0$.

Not all divisors are existent in every GLSM phase. We have seen for example that in the orbifold phase of $\mathbb{C}^3/\mathbb{Z}_3$, the D-term constraint forces the exceptional coordinate x to be nonzero. Hence the exceptional divisor $E := \{x = 0\}$ can not exist in this phase.

Two divisors are said to be linear equivalent if all their topological quantities are equal. In [10] and [9], the equivalence relations between D_i and E for $\mathbb{C}^3/\mathbb{Z}_3$ are given by

$$E \sim -3D_i, \qquad D_i \sim D_j. \tag{2.41}$$

Using Poincaré duality, one can show that linear equivalent divisors have the same intersection number with any arbitrary curve C, i.e.

$$\text{if} \quad A \sim B \quad \text{then} \quad AC = BC, \tag{2.42}$$

in which with AC we mean the intersection number of divisor A with the curve C. Such equivalence relations will be a very important tool to compute triple intersection numbers of divisors lateron.

To clarify the notation, we have to note that the multiplication of a divisor A with a scalar n means nothing other than it has the same intersection numbers with all curves C as A multiplied with n, so that we can write for any curve C

$$\text{if} \qquad B \sim n \cdot A \qquad \text{then} \qquad BC = n \cdot AC. \tag{2.43}$$

2.2.3 Fully resolvable GLSMs

A toroidal orbifold in general can contain more than just one orbifold fixed point. So generally, there are often more than one exceptional gauging possible to resolve all orbifold singularities. The resolution procedure where the maximum number of exceptional gaugings are performed is called the maximal fully resolvable model. In [7] it has been used to resolve $T^6/\mathbb{Z}_3$, $T^6/\mathbb{Z}_4$ and $T^6/\mathbb{Z}_{6-II}$.

The other extremal case is the so-called minimal fully resolvable model, where only a minimal number of exceptional gaugings is introduced to resolve all orbifold singularities. Since the number of D- and F-term constraints increase with the number of exceptional gaugings, the minimal fully resolvable model might be a bit more handy because it contains less FI-parameters than the maximal fully resolvable model. For example, in $T^6/\mathbb{Z}_3$, only one gauging is required to resolve all its 27 orbifold singularities at once. The minimal fully resolvable model was first stated in [11].

Of course, there are often also fully resolvable models with less exceptional gaugings as in the maximal fully resolvable model but more as in the minimal one. However, it is a very important and quite surprising fact that a model with the same gauging as in the minimal fully resolvable model plus some additional gaugings normally is not fully resolvable anymore. One always has to check if all orbifold singularities vanish in the blow-up phase. If only some of them are resolved, the model is called partially resolvable. We will restrict ourselves to fully resolvable models in the rest of this work.

2.2.4 Phase structure

As we have already seen in the resolution of $\mathbb{C}^3/\mathbb{Z}_3$, GLSM orbifold resolutions possess a certain phase structure depending on which values the corresponding FI-parameters may take. The effect of a transition between such phases is a rather big one: the target space topology changes. For some values the target space might be a singular orbifold whereas for others it is a resolved smooth Calabi-Yau. There is even a phase where the F-term contraints force all coordinates z to be zero, hence the target space collapses to a point there. This is a major advantage of the GLSM resolution procedure since it describes these phase transitions smoothly by varying the associated parameters.

One can classify the effects of a phase transition in the following three groups [7]:

1. The intersection set of two divisors may change. This of course also means that triple intersection numbers can change.

2. Divisors may be existent in certain phases, whereas in others they vanish. For example, the exceptional divisor of $\mathbb{C}^3/\mathbb{Z}_3$ is existent in the blow-up phase, but has to vanish in the orbifold phase, since x has to have a VEV there.

3. Even the target space dimension can be modified. As already said, we can have an orbifold in one phase which collapses to a point in another.

We will compute the triple intersection numbers of divisors in $T^6/\mathbb{Z}_3$ in various phases in a subsequent section.

2.2.5 Torus lattices

A torus lattice is described by a set of vectors which realize the torus symmetries of an orbifold. For example, a two-torus with the identifications $u \sim u+1 \sim u+\tau$ with some complex structure τ has the $\mathbb{R}^2$ basis vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}Re(\tau)\\Im(\tau)\end{pmatrix}$. Hence points differing by an integer combination of lattice vectors are identified.

A six-dimensional torus lattice for $u_a \sim u_a+1 \sim u_a+\tau_a$ can be described analogously. It is factorized to $A_2 \times A_2 \times A_2$ if there is no other action than the orbifold symmetry θ acting on more than one of the two-tori T_2 spanned by each coordinate u_a. In contrast, there are resolution models for $T^6/\mathbb{Z}_3$ that involve 3-volution symmetries that act in more than just one of such two-tori simultaneously. This leads to non-factorized lattices such as $F_4 \times A_2$ or E_6. In Appendix B of [7] it is shown that one can factorize these lattices to A_2^3 again. Lateron, we will compute the intersection number of $R_1R_2R_3$ of $T^6/\mathbb{Z}_3$ on a non-factorized E_6 lattice.

Chapter 3

Orbifold resolutions and intersection numbers with toric geometry

Before we start the main part of this work, namely the GLSM resolution of $T^6/\mathbb{Z}_3$ and the computation of intersection numbers of divisors living therein, we take a look at the work of [9]. There, toric geometry was used to resolve orbifold singularities and to compute intersection numbers. In particular, we will exemplarily show how to resolve orbifold singularities that locally look like $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_{6-I}$. Afterwards the nonvanishing intersection numbers of the $T^6/\mathbb{Z}_3$ orbifold will be computed explicitly. It is the main aim of the following chapters to reproduce the intersection numbers that were found in [9].

3.1 Toric varieties

To begin with, we need to have a definition of what is called a toric variety.

Definition 1 (Toric variety) *A toric variety X is a complex algebraic variety containing an algebraic torus $T = (\mathbb{C}^*)^r$ as a dense open set, together with an action of T on X whose restriction to $T \subset X$ is just the usual multiplication on T [12].*

Technically, this means that a toric variety can be described as

$$X_\Sigma = (\mathbb{C}^d \backslash F_\Sigma) / (\mathbb{C}^*)^r\,, \tag{3.1}$$

which says that this is the d-dimensional complex space with r independent $\mathbb{C}^*$ symmetries. For the toric variety to be well defined, we have to exclude the set F_Σ of points that are fixed under any continuous subgroup of $(\mathbb{C}^*)^r$. F_Σ is therefore called the fixed point set.

A toric variety can be entirely described by an auxiliary object called a *fan*. To give the definition of a fan, we need to introduce a lattice $N \cong \mathbb{Z}^{d-r}$. We call $N_{\mathbb{R}} = N \otimes \mathbb{R}$.

Definition 2 (Cone) *A strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$ is defined as the set*

$$\sigma = \{a_1 v_1 + a_2 v_2 + ... + a_k v_k | a_i \geq 0\} \tag{3.2}$$

with a finite set of vectors $v_1, ..., v_k$ and $\sigma \cap (-\sigma) = \{0\}$ (strong convexity) [14].

From now on, "strongly convex rational polyhedral cone" will simply be abbreviated by "cone".

Definition 3 (Fan) *A fan Σ is a set of cones in $N_{\mathbb{R}}$ which satisfy the conditions*

1. *each face of a cone in Σ is also a cone in Σ,*
2. *the intersection of two cones in Σ is a face of each.*

To get a feeling for these definitions, let us see a simple example:

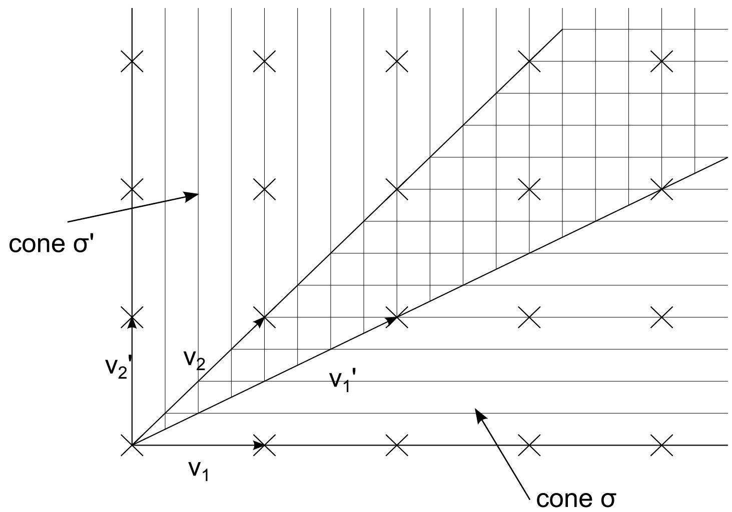

Figure 3.1: Two cones σ and σ' generated by v_1, v_2 and v_1', v_2' in a lattice $N \cong \mathbb{Z}^2$

If the cone $\sigma = \{a_1 v_1 + a_2 v_2 | a_i \geq 0\}$ is a cone in Σ,

- so are the cones $\sigma_1 = \{b_1 v_1 | b_1 \geq 0\}$ and $\sigma_2 = \{b_2 v_2 | b_2 \geq 0\}$ (rule 1).

- the cone σ' can not be in Σ because the intersection $\sigma \cap \sigma'$ is not a face of each cone (rule 2).

Note that every k - dimensional cone $\sigma \in \Sigma$ generated by $v_1, ..., v_k$ is associated to the codimension k subvariety

$$X_\sigma = \{z \in X_\Sigma | z_1 = ... = z_k = 0\}. \tag{3.3}$$

In particular, the generators v_i of the fan Σ correspond to the set of divisors in X_Σ. Furthermore, there is a correspondence between v_i and the homogeneous coordinate z_i.

To find the components of the vectors v_i, we look at

$$\Phi : \mathbb{C}^d \to \mathbb{C}^n : (z_1, ..., z_d) \to (\prod_{i=1}^{d} z_i^{v_i^1}, ..., \prod_{i=1}^{d} z_i^{v_i^n}), \tag{3.4}$$

with v_i^j denoting the j-th component of the vector v_i. This defines a map from the covering space $\mathbb{C}^d \backslash F_\Sigma$ to the toric variety X_Σ. Clearly, such a map has to be invariant under the $(\mathbb{C}^*)^r$ symmetry of X_Σ. This means that a $(\mathbb{C}^*)^r$ symmetry like

$$(z_1, ..., z_n) \sim (\lambda_a^{Q_1^{(a)}} z_1, ..., \lambda_a^{Q_n^{(a)}} z_n), \qquad a = 1, ..., r \tag{3.5}$$

generically leads to

$$\sum_i Q_i^{(a)} v_i^k = 0. \tag{3.6}$$

Discrete orbifold symmetry groups G generated by

$$\theta : (z_1, ..., z_n) \sim (\epsilon z_1, \epsilon^{p_1} z_2, ..., \epsilon^{p_{n-1}} z_n), \qquad \epsilon = e^{2\pi i/p} \tag{3.7}$$

are analogously translated to

$$v_1^k + p_1 v_2^k + ... + p_{n-1} v_n^k = 0 \mod \text{p} \tag{3.8}$$

3.2 Local resolutions of orbifold singularities

A fixed point of a three-dimensional orbifold locally looks like $\mathbb{C}^3/G$, with some finite group G. With the means of toric geometry, such a singularity can be resolved obtaining a toric variety like (3.1). The corresponding resolution process is described in this section.

For $\mathbb{C}^3/G$ singularities, (3.8) becomes

$$v_1^k + p_1 v_2^k + p_2 v_3^k = 0 \mod \text{p}.$$

From [9] we know that the last component of every vector v_i in Σ has to be 1 except of the vector $v_0 = (0, 0, 0)$. This implies that X_Σ has trivial canonical class and is Calabi-Yau [9]. One can show that X_Σ is smooth if all top-dimensional cones in Σ have volume 1. In the unresolved orbifold, however, there is just one top-dimensional cone spanned by v_1, v_2, v_3 which has a volume of $|G|$, so the orbifold is singular.

This is where the resolution process comes into play: we add to the generators of the fan Σ all the lattice points w_i that lie in between the points v_1, v_2, v_3 and fulfill the Calabi - Yau condition that the last component of each vector has to be 1, i.e. $w_i^3 = 1$. To these generators w_i, we associate the exceptional coordinate y_i and the corresponding exceptional divisor E_i. All the possible top-dimensional cones of the resulting fan $\tilde{\Sigma}$ now have volume 1, hence the resulting toric variety $X_{\tilde{\Sigma}}$ is smooth. Since there are now more than three generators at our disposal, there are often multiple possibilities to form the resolved fan $\tilde{\Sigma}$, i.e. to construct a set of cones out of the generators that fulfill the fan conditions of definition 3. This ambiguity in forming the resolved fan is associated to different possible triangulations in the toric diagram. These various possibilities only differ in their exclusion set $F_{\tilde{\Sigma}}$ which is obtained as follows: Take all v_i, w_j that do not span a cone in $\tilde{\Sigma}$. Then all points that have zeros in the associated coordinates $(z_i, y_j) = 0$ belong to the exclusion set $F_{\tilde{\Sigma}}$ of the toric variety $X_{\tilde{\Sigma}}$.

In the GLSM language, the exclusion set F_Σ translates to some FI-parameter $b > 0$ of a blown-up orbifold that excludes zeros of some set of coordinates by the corresponding D-term constraint. For example, in $\mathbb{C}^3/\mathbb{Z}_3$ the only existing D-term constraint is

$$|z_1|^2 + |z_2|^2 + |z_3|^2 - 3|x|^2 = b. \tag{3.9}$$

So obviously, in the blow-up phase where $b > 0$, the coordinates z_1, z_2, z_3 can not vanish all together. Hence the point $(z_1, z_2, z_3) = (0, 0, 0)$ forms the exclusion set F_Σ of $\mathbb{C}^3/\mathbb{Z}_3$. As we will see, the corresponding generators v_1, v_2, v_3 indeed never share a cone in the resolution of $\mathbb{C}^3/\mathbb{Z}_3$.

3.2.1 The Mori-cone

The relevant data of a toric variety can be compactly summarized using a so-called $(P|Q)$-Matrix, in which the rows of P are the vectors v_i, w_j and the a-th column of Q are the $Q_i^{(a)}$ from (3.5). Obviously, there is still an ambiguity left on the vectors $Q^{(a)}$. To visualize this, take for example a $(\mathbb{C}^*)^2$ - action that acts as

$$(z_1, z_2, z_3, y_1) \sim (\lambda_1 z_1, \lambda_2 z_2, \lambda_1\lambda_2 z_3, \frac{1}{\lambda_1^2\lambda_2^2} y_1).$$

This translates to the vectors $Q^{(1)} = (1, 0, 1, -2)$ and $Q^{(2)} = (0, 1, 1, -2)$. Exact the same $(\mathbb{C}^*)^2$ symmetry can be described by renaming for example

$\lambda'_1 = \lambda_1\lambda_2$ and $\lambda'_2 = \lambda_2$. The symmetry action then would look like

$$(z_1, z_2, z_3, y_1) \sim (\frac{\lambda'_1}{\lambda'_2} z_1, \lambda'_2 z_2, \lambda'_1 z_3, \frac{1}{\lambda'^2_1} y_1)$$

which results in $Q^{(1)} = (1, 0, 1, -2)$ and $Q^{(2)} = (-1, 1, 0, 0)$.

It is convenient to choose the vectors $Q^{(a)}$ to be the generators of the Mori-cone C_a. The Mori-cone is the space of all curves $C \in X_\Sigma$ that have positive intersection numbers with all divisors $D \in X_\Sigma$. A useful result is that the components of the generators of the Mori-cone directly give the intersection number of the curve C_a with the corresponding divisor. The whole derivation of the construction of the Mori-cone is given in [15], but [9] presents a nice and short recipe of how to construct it in this context:

1. In a specific triangulation, consider all top-dimensional cones S_k of the fan $\tilde{\Sigma}$. Take all pairs of top-dimensional cones (S_k, S_l) that have two generators in common.

2. Find the minimal integer relation between the four generators of each such pair (S_k, S_l). The coefficients of the complementary generators $(S_k \cup S_l) \backslash (S_k \cap S_l)$ have to be positive.

3. Find the minimal integer basis of the equations obtained in 2. The resulting coefficients are the components of the generators of the Mori-cone.

This procedure will surely become clear when we do our first example.

Equivalence relations between the divisors can be obtained using the intersection numbers of the divisors with the Mori-curves C_a. Two divisors are equivalent if they have the same intersection numbers with all Mori-curves C_a. The intersection number of three distinct divisors is one if the three divisors share a cone in $\tilde{\Sigma}$ and is zero otherwise.

3.3 Examples

Now we have got all tools at hand to locally resolve orbifold singularities and to compute the intersection numbers of the appearing divisors. We will now explicitly resolve the singularities of $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_{6-I}$.

3.3.1 Resolution of $\mathbb{C}^3/\mathbb{Z}_3$

The group action θ of $\mathbb{Z}_3$ on $\mathbb{C}^3$ reads

$$\theta : (z_1, z_2, z_3) \to (\zeta z_1, \zeta z_2, \zeta z_3), \qquad \epsilon = \zeta = e^{2\pi i/3}. \tag{3.10}$$

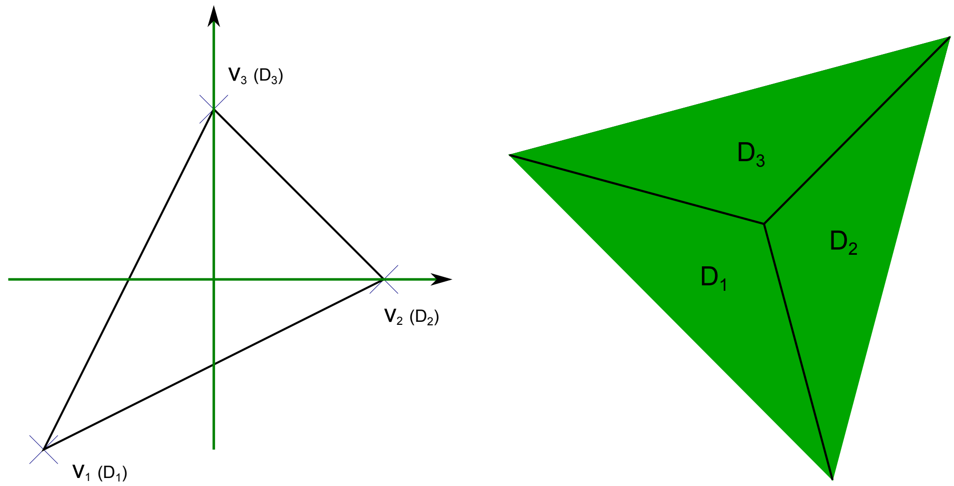

Figure 3.2: Toric diagram of $\mathbb{C}^3/\mathbb{Z}_3$ and the corresponding dual graph

Thus, (3.8) becomes $v_1^k + v_2^k + v_3^k = 0 \mod 3$. In accordance with this equation, the generators of the unresolved fan Σ can be chosen to be

$$v_1 = (-1,-1,1), \qquad v_2 = (1,0,1), \qquad v_3 = (0,1,1). \tag{3.11}$$

The corresponding toric diagram is obtained by simply plotting these points in the hyperplane $(x, y, 1)$. A connecting line of two such points in the toric diagram then corresponds to the intersection curve of the associated divisors and a face bordered by the connecting lines of three divisors represents the intersection point of these three divisors. The toric diagram of $\mathbb{C}^3/\mathbb{Z}_3$ and its dual graph are shown in figure 3.2.

In general, the dual toric diagram is more intuitive to read because intersection points of three divisors are also points in the dual diagram, whereas intersection curves of two divisors correspond to curves and divisors correspond to faces. We can get the dual diagram by simply requiring its properties. This means we just have to change points into faces, lines into lines and faces into points, preserving the combinatorial data of the original toric diagram.

To resolve the singularity, we see that there is only one point in between the points v_i that we have to add to the generators of the fan to get a smooth toric variety. Namely, this point is

$$w = (0,0,1). \tag{3.12}$$

Obviously, there is only one possible triangulation. The toric diagram of the resolved singularity and its dual graph are plotted in figure 3.3. We directly see that there are three top-dimensional cones possible, that is to

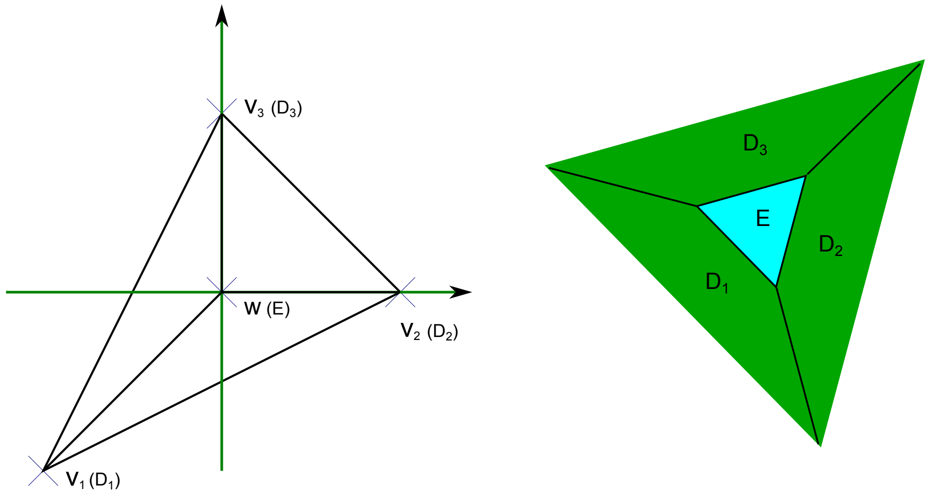

Figure 3.3: Toric diagram of the resolution of $\mathbb{C}^3/\mathbb{Z}_3$ and the corresponding dual graph

say the cones generated by

$$S_1 = (v_1, v_2, w), \quad S_2 = (v_2, v_3, w), \quad S_3 = (v_3, v_1, w). \tag{3.13}$$

To find the Mori-cone, we proceed as described in the presented recipe given in the previous section.

1. *Take all pairs of top-dimensional cones that have two generators in common:*

   All the three cones S_1, S_2, S_3 share two generators with the residual two cones:

$$\begin{aligned} S_1 \cup S_2 &= \{v_1, v_2, v_3, w\}, \\ S_1 \cup S_3 &= \{v_1, v_2, v_3, w\}, \\ S_2 \cup S_3 &= \{v_1, v_2, v_3, w\}. \end{aligned}$$

2. *Find the minimal integer relation between the generators of each such pair. The coefficients of the complementary elements $(S_k \cup S_l) \backslash (S_k \cap S_l)$ have to be positive.*

   Since all pairs of cones in 1. form the same set of generators, they trivially lead all to the same relation

$$v_1 + v_2 + v_3 - 3w = 0.$$

3. *Find the minimal integer basis of the equations obtained in 2. The coefficients of each equation of this basis encode the components of the corresponding generator of the Mori-cone.*

In this case, there is only one equation in 2. Hence the wanted basis is the equation itself. This leads to the generator of the Mori-cone

$$C = (1, 1, 1, -3). \tag{3.14}$$

Since the components of C directly give the intersection numbers of the curve defined by C with the corresponding divisors, we can easily read off the equivalence relations:

$$D_i \sim D_j, \qquad E \sim -3D_i. \tag{3.15}$$

Therefore, the curve defined by C can be written as

$$C = D_1E = D_2E = D_3E. \tag{3.16}$$

With this information at hand it is not hard to determine the possible intersection numbers:

$$E^3 = 9, \qquad D_iE^2 = -3, \qquad D_iD_jE = 1, \qquad D_iD_jD_k = -\frac{1}{3}. \tag{3.17}$$

The $(P|Q)$ matrix is in this case

$$(P|Q) = \begin{pmatrix} -1 & -1 & 1 & | & 1 \\ 1 & 0 & 1 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & -3 \end{pmatrix}. \tag{3.18}$$

The generator of the Mori-cone also encodes the $\mathbb{C}^*$ symmetry acting on the resolved toric variety. It is

$$(z_1, z_2, z_3, y) \sim (\lambda z_1, \lambda z_2, \lambda z_3, \frac{y}{\lambda^3}) \tag{3.19}$$

and the new blown up geometry reads

$$X_{\tilde{\Sigma}} = (\mathbb{C}^4 \backslash \{0\})/\mathbb{C}^*. \tag{3.20}$$

3.3.2 Resolution of $\mathbb{C}^3/\mathbb{Z}_{6-I}$

The $\mathbb{Z}_{6-I}$ action θ on $\mathbb{C}^3$ is

$$\theta : (z_1, z_2, z_3) \sim (\kappa z_1, \kappa z_2, \kappa^4 z_3) \qquad \epsilon = \kappa = e^{2\pi i/6}. \tag{3.21}$$

This directly leads to $v_1^k + v_2^k + 4v_3^k = 0 \mod 6$. A convenient choice of the generators of the fan Σ is

$$v_1 = (1, -2, 1), \qquad v_2 = (-1, -2, 1), \qquad v_3 = (0, 1, 1). \tag{3.22}$$

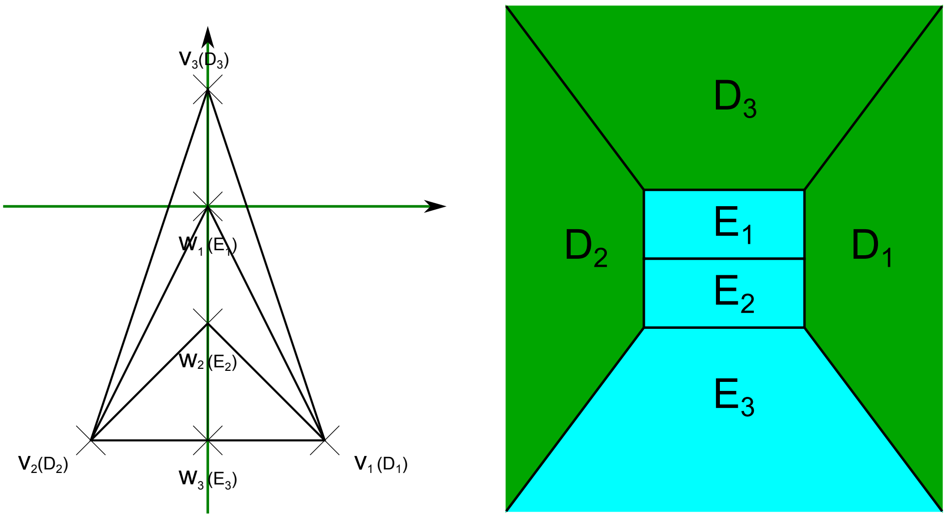

Figure 3.4: Toric diagram of $\mathbb{C}^3/\mathbb{Z}_{6-I}$ and its dual graph

In figure 3.4 we see that we have to add the generators

$$w_1 = (0,0,1), \qquad w_2 = (0,-1,1), \qquad w_3 = (0,-2,1). \tag{3.23}$$

in order to only have volume 1 top-dimensional cones in the resolved fan $\tilde{\Sigma}$, which leads to a smooth toric variety $X_{\tilde{\Sigma}}$. As can be seen in the toric diagram, the triangulation is unique again.

We see that the generators $(v_3, w_2), (v_3, w_3), (w_1, w_3)$ and (v_1, v_2) never share a cone in $\tilde{\Sigma}$. This means that the exclusion set $F_{\tilde{\Sigma}}$ is

$$F_{\tilde{\Sigma}} = \{(z_3, y_2) = 0, (z_3, y_3) = 0, (y_1, y_3) = 0, (z_1, z_2) = 0\}. \tag{3.24}$$

To determine the Mori-cone, we have to look which 3-dimensional cones are contained in $\tilde{\Sigma}$. These are the cones spanned by $S_1 = (v_1, y_2, y_3)$, $S_2 = (v_1, y_2, y_1)$, $S_3 = (v_1, y_1, z_3)$, $S_4 = (v_2, y_2, y_3)$, $S_5 = (v_2, y_2, y_1)$ and $S_6 = (v_2, y_1, z_3)$.

The pairs of cones having two generators in common and the minimal integer relations between the corresponding generators are

$$\begin{aligned}
S_6 \cup S_3 &= \{v_1, v_2, v_3, w_1\}, & v_1 + v_2 + 4v_3 - 6w_1 &= 0,\\
S_5 \cup S_2 &= \{v_1, v_2, w_1, w_2\}, & v_1 + v_2 + 2w_1 - 4w_2 &= 0,\\
S_4 \cup S_1 &= \{v_1, v_2, w_2, w_3\}, & v_1 + v_2 - 3w_3 &= 0,\\
S_3 \cup S_2 &= \{v_1, v_3, w_1, w_2\}, & v_3 - 2w_1 + w_2 &= 0,\\
S_2 \cup S_1 &= \{v_1, w_1, w_2, w_3\}, & w_1 - 2w_2 + w_3 &= 0,\\
S_6 \cup S_5 &= \{v_2, v_3, w_1, w_2\}, & v_3 - 2w_1 + w_2 &= 0,\\
S_5 \cup S_4 &= \{v_2, w_1, w_2, w_3\}, & w_1 - 2w_2 + w_3 &= 0.
\end{aligned}$$

The minimal integer basis of these 7 equations is

$$v_1 + v_2 - 2w_3 = 0, \quad v_3 - 2w_1 + w_2 = 0, \quad w_1 - 2w_2 + w_3 = 0.$$

This directly translates into the Mori-cone which is spanned by

$$\begin{aligned} C_1 &= (1,1,0,0,0,-2), \\ C_2 &= (0,0,1,-2,1,0), \\ C_3 &= (0,0,0,1,-2,1). \end{aligned} \tag{3.25}$$

To see to which intersection curves these generators belong, we notice that the generators associated to the divisors D_1D_2, E_1E_3, D_3E_3, and D_3E_2 never span a cone in $\tilde{\Sigma}$, hence all triple intersections containing these pairs vanish. Since the intersection of C_1 with D_1, D_2 and E_3 is nonzero, C_1 can not be an intersection of one of the divisors of D_1, D_2, D_3, E_1 with any other divisor. Hence, $C_1 = E_2E_3$. Analogously one can see that $C_2 = D_1E_1 = D_2E_1$ and $C_3 = E_2D_1 = E_2D_2$.

Divisors that have the same intersection number with all of these three curves are equivalent. Thus we can read of the equivalence relations

$$\begin{aligned} 0 &\sim 6D_1 + E_1 + 2E_2 + 3E_3, \\ 0 &\sim 6D_2 + E_1 + 2E_2 + 3E_3, \\ 0 &\sim 3D_3 + 2E_1 + E_2. \end{aligned} \tag{3.26}$$

The $(\mathbb{C}^*)^3$ symmetry of the resolved orbifold singularity can be read off from (3.25):

$$(z_1, z_2, z_3, y_1, y_2, y_3) \sim (\lambda_1 z_1, \lambda_1 z_2, \lambda_2 z_3, \frac{\lambda_3}{\lambda_2^2} y_1, \frac{\lambda_2}{\lambda_3^2} y_2, \frac{\lambda_3}{\lambda_1^2} y_3). \tag{3.27}$$

The resolved geometry is

$$X_{\tilde{\Sigma}} = (\mathbb{C}^6 \backslash F_{\tilde{\Sigma}})/(\mathbb{C}^*)^3. \tag{3.28}$$

3.4 Global orbifold resolutions

3.4.1 Inherited divisors

In a compact orbifold, there exist further divisors that can have nonvanishing intersection numbers with the already familiar divisors $D_{i\alpha}$ which fix the covering space coordinate z_i to an orbifold fixed point $z_i = z_{fixed,i,\alpha}$, and the exceptional divisors E. These divisors are called inherited divisors R_i and can be defined by fixing the local orbifold coordinate $\tilde{z}_i$ to a constant value $\tilde{z}_i = c^{n_i} \neq \tilde{z}_{fixed,i,\alpha}$, where n_i is the order of the orbifold action in the i-th T^2-torus. Here we have to be careful not to confuse the local orbifold coordinate $\tilde{z}_i$ with the coordinate z_i on the cover. With a convenient choice of cover coordinates the inherited divisor R_i can be defined as [9]

$$R_i := \bigcup_{k=1}^{n_i} \left\{ z_i = \epsilon^k c \right\}. \tag{3.29}$$

Multiplying all these equations together we obtain $z_i^{n_i} = c^{n_i} = \tilde{z}_i$. Keeping in mind that the ordinary divisors $D_{i\alpha}$ are defined by fixing the cover coordinate z_i to a constant value whereas the inherited divisors fix $\tilde{z}_i = z_i^{n_i}$, one can claim the equivalence relation

$$R_i \sim n_i D_{i,\alpha}, \tag{3.30}$$

which is valid near the orbifold fixed point.

In the resolved orbifold, the equivalence relation is assumed to take the form [9]

$$R_i \sim n_i D_{i,\alpha} + \sum_{k,\beta,\gamma} a_{k,\alpha,\beta,\gamma} E_{k,\alpha,\beta,\gamma}, \qquad \forall \alpha, i, \tag{3.31}$$

where $E_{k,\alpha,\beta,\gamma}$ denotes the exceptional divisors emerging from the resolution process with some coefficients $a_{k,\alpha,\beta,\gamma}$ depending on the singularities we deal with.

Furthermore, one can define inherited divisors that do not fix the two degrees of freedom of one complex coordinate z_i, but fix one degree of freedom of two distinct coordinates z_i, z_j, $i \neq j$. This is only possible if the order of the orbifold action in both coordinates is equal, $n_i = n_j = n$. This divisor is defined as

$$R_{ij} = \bigcup_{k=1}^{n} \{ z_k^{ij} + \bar{z}_k^{ij} = c^{ij} \} \cup \{ z_k^{ij} - \bar{z}_k^{ij} = c^{ij} \} \tag{3.32}$$

with $z_k^{ij} = z_i + \epsilon^k z_j$. It is not known how to determine the corresponding equivalence relations (3.31) explicitly, but they also are claimed to be [9]

$$R_{ij} \sim n D_{ij,\alpha} + \sum_{k,\beta,\gamma} a_{k,\alpha,\beta,\gamma} E_{k,\alpha,\beta,\gamma}. \tag{3.33}$$

3.4.2 Intersection numbers involving inherited divisors

In many cases, there is a very direct way to compute the intersection numbers between the inherited divisors R_i and the ordinary divisors $D_{j,\alpha}$ using their definition polynomials on the cover. For example, consider $R_1 R_2 R_3$. These divisors are defined on the covering space as $R_i := \{z_i^{n_i} = c_i^{n_i}\}$. So, $R_1 R_2 R_3$ is simply the number of solutions of the equation system $\{z_1^{n_1} = c_1^{n_1}, z_2^{n_2} = c_2^{n_2}, z_3^{n_3} = c_3^{n_3}\}$, which clearly is $n_1 n_2 n_3$. But, since this is on the covering space, we have not taken into account that the orbifold action θ identifies these solutions in groups of size $|G|$, by which we still have to divide. This finally yields

$$R_1 R_2 R_3 = \frac{n_1 n_2 n_3}{|G|}. \tag{3.34}$$

Since the ordinary divisors $D_{i,\alpha}$ are defined linearly on the cover, intersection numbers involving them are simply obtained setting the corresponding n_i equal to 1. Hence

$$R_i R_j D_{k,\alpha} = \frac{n_i n_j}{|G|}, \qquad R_i D_{j,\alpha} D_{k,\beta} = \frac{n_i}{|G|}. \tag{3.35}$$

Note that R_i and $D_{i,\alpha}$ can never intersect as well as R_i does not self-intersect. Thus, i, j, k have to be pairwise distinct.

Using the definition (3.32), one also can find such equations involving the divisors R_{ij}. Some of them are for example

$$R_{ij} R_{ji} R_k = -\frac{n_i^2 n_k}{|G|}, \qquad R_{ij} R_{jk} R_{ki} = \frac{n_i^3}{|G|}, \tag{3.36}$$

where the negative sign is an effect of taking into account the change of orientation of complex conjugation. Since R_{ij} can only exist if $n_i = n_j$, we can set $n_j = n_i$ in the first and $n_k = n_j = n_i$ in the second intersection number of (3.36).

3.4.3 The resolution process

The global resolution in [9] was done as follows: As in the local resolutions, a lattice $N \cong \mathbb{Z}^3$ has to be introduced. The basis of this lattice is this time $f_i = m_i e_i$ with e_i the euclidean basis vectors. The m_i are $\in \mathbb{N}^+$ and must be chosen such that they satisfy $m_1 m_2 m_3 = n_1 n_2 n_3 / |G|$. Then the inherited divisors correspond to the one-dimensional cones

$$R_i : \quad v_i = -f_i, \qquad i = 1, 2, 3. \tag{3.37}$$

The ordinary divisors D_i are associated to the vectors

$$v_{i+3} = n_i f_i, \qquad i = 1, 2, 3 \tag{3.38}$$

and the exceptional divisors E_i are again the lattice points that lie in between the triangle spanned by the points $\{v_4, v_5, v_6\}$. Roughly speaking, this procedure continues like in the resolution of local singularities. This means we look for possible triangulations that fulfill the fan conditions and use the basic rule that the intersection number is 1 if the relevant divisors share a 3-dim cone in the resolved fan $\tilde{\Sigma}$ and is 0 else.

3.4.4 Nonvanishing intersection numbers in $T^6/\mathbb{Z}_3$

Fortunately, in this example, we can go the easy way computing the intersection numbers using the equations (3.34)-(3.36). Without much effort we see that

$$\begin{aligned} &R_1R_2R_3 = R_{12}R_{23}R_{31} = R_{13}R_{32}R_{21} = 9, \\ &R_1R_{23}R_{32} = R_2R_{13}R_{31} = R_3R_{12}R_{21} = -9. \end{aligned} \tag{3.39}$$

Using the equivalence relation

$$R_i \sim 3D_{i,\alpha} + \sum_{\beta,\gamma=1}^{3} E_{\alpha,\beta,\gamma}, \tag{3.40}$$

we get the only nonvanishing intersection number left, namely

$$E^3_{\alpha,\beta,\gamma} = 9. \tag{3.41}$$

Chapter 4

GLSM resolutions of $T^6/\mathbb{Z}_3$ orbifolds

The $T^6/\mathbb{Z}_3$ orbifold is one of the easiest and best understood orbifolds, which is the main reason why we focus on this orbifold here. We review the maximal fully resolvable model for this orbifold from [7] as well as the computation of the intersection numbers in this resolution model. Afterwards, we take a look at the minimal fully resolvable model and determine the intersection numbers based on this resolution model, which has not been done before in [7]. Moreover, we consider further resolution models which lead to factorized A_2^3 lattices before we examine a resolution model leading to a non-factorized E_6 lattice. To this model, we depict remaining problems we had in the computation of the intersection numbers.

We use the same notation as in [7], in particular we label the 27 fixed points of $T^6/\mathbb{Z}_3$ by the indices (α, β, γ) each running from 1 to 3, denoting the the place of the singularity in the first, second and third two-torus, respectively.

To construct a six-torus with a $\mathbb{Z}_3$ orbifold symmetry out of our GLSM, we introduce the superpotential W_{torus},

$$W_{\text{torus}} = \sum_{a,i} \mathcal{C}_a \mathcal{Z}_{ai}^3 \tag{4.1}$$

with the chiral superfields $\mathcal{C}_a$ and $\mathcal{Z}_{ai}$ with the charge assignment given in table 4.1.

The F_{c_a}-term constraints then directly give the torus defining equations (2.17) in Weierstraß-mapped coordinates,

$$z_{a1}^3 + z_{a2}^3 + z_{a3}^3 = 0.$$

As we have already seen, the orbifold-action θ is

$$\theta : (z_{11}, z_{21}, z_{31}) \sim (\zeta z_{11}, \zeta z_{21}, \zeta z_{31}),$$

Superfield U(1) charges\ scalar component	$\mathcal{Z}_{1i}$ z_{1i}	$\mathcal{Z}_{2j}$ z_{2j}	$\mathcal{Z}_{3k}$ z_{3k}	$\mathcal{C}_1$ c_1	$\mathcal{C}_2$ c_2	$\mathcal{C}_3$ c_3
R_1	1	0	0	-3	0	0
R_2	0	1	0	0	-3	0
R_3	0	0	1	0	0	-3

Table 4.1: Charge assignment for the construction of a T^6-torus possessing a $\mathbb{Z}_3$ symmetry

whereas the 3-volutions α_a act as

$$\alpha_a : (z_{a1}, z_{a2}, z_{a3}) \sim (z_{a1}, \zeta^2 z_{a2}, \zeta z_{a3}).$$

An exceptional gauging $U(1)_{E_{\alpha\beta\gamma}}$ spares a $\mathbb{Z}_3$-action on the coordinates that looks like

$$\theta_{\alpha\beta\gamma} : (z_{1\alpha}, z_{2\beta}, z_{3\gamma}) \sim (\zeta z_{1\alpha}, \zeta z_{2\beta}, \zeta z_{3\gamma}). \tag{4.2}$$

It can be shown that these are not $3^3 = 27$ but only 4 independent $\mathbb{Z}_3$ actions that can be generated by $\theta, \alpha_1, \alpha_2$ and α_3 as

$$\theta_{\alpha\beta\gamma} = \theta \alpha_1^{\alpha-1} \alpha_2^{\beta-1} \alpha_3^{\gamma-1}. \tag{4.3}$$

4.1 The maximal fully resolvable model

In the maximal fully resolvable model, an exceptional gauging $U(1)_{E_{\alpha\beta\gamma}}$ is introduced for each of the 27 $T^6/\mathbb{Z}_3$ orbifold fixed points. Since each $T^6/\mathbb{Z}_3$ fixed point locally looks like a $\mathbb{C}^3/\mathbb{Z}_3$ singularity, we recall the corresponding GLSM resolution charge assignment, 2.1. Together with the torus-constructing fields and charges given in table 4.1, we can claim the overall charge assignment for $T^6/\mathbb{Z}_3$ in the maximal model by table 4.2. Performing our exceptional gaugings, we have to take care not to modify the target space dimension. For this reason we again introduce the exceptional coordinates $x_{ijk},\ i,j,k = 1,...,3$, one for each exceptional gauging. As we have seen in section 2.2.1, if x_{ijk} takes on a VEV, this leads to a $\mathbb{Z}_3$ symmetry acting on the coordinates (z_{1i}, z_{2j}, z_{3k}). This is the reason to claim equation (4.2). We see that in the maximal fully resolvable model, we have four independent $\mathbb{Z}_3$-actions $\mathbb{Z}_{3,\text{orbi}} \times \mathbb{Z}^3_{3,3\text{-vol}}$, namely θ, α_1, α_2 and α_3.

The D-term constraints can easily be read off from the charge assignment:

$$|z_{a1}|^2 + |z_{a2}|^2 + |z_{a3}|^2 - 3|c_a|^2 = a_a, \tag{4.4}$$
$$|z_{1\alpha}|^2 + |z_{2\beta}|^2 + |z_{3\gamma}|^2 - 3|x_{\alpha\beta\gamma}|^2 = b_{\alpha\beta\gamma}, \tag{4.5}$$

Superfield U(1) charges\ s. comp.	$\mathcal{Z}_{1i}$ z_{1i}	$\mathcal{Z}_{2j}$ z_{2j}	$\mathcal{Z}_{3k}$ z_{3k}	$\mathcal{C}_1$ c_1	$\mathcal{C}_2$ c_2	$\mathcal{C}_3$ c_3	$\mathcal{X}_{ijk}$ x_{ijk}
R_1	1	0	0	-3	0	0	0
R_2	0	1	0	0	-3	0	0
R_3	0	0	1	0	0	-3	0
$E_{\alpha\beta\gamma}$	$\delta_{i\alpha}$	$\delta_{j\beta}$	$\delta_{k\gamma}$	0	0	0	-3 $\delta_{i\alpha}\delta_{j\beta}\delta_{k\gamma}$

Table 4.2: Charge assignment for $T^6/\mathbb{Z}_3$ in the maximal fully resolvable model

with a, α, β and γ running from 1 to 3.

Since we performed the exceptional gaugings $U(1)_{E_{\alpha\beta\gamma}}$, we have lost the gauge invariance of our six-torus producing potential (4.1). This can be fixed by multiplying the exceptional fields $\mathcal{X}_{ijk}$ to the fields $\mathcal{Z}_{ai}$ in a proper way. Equation (4.1) then becomes

$$W_{\text{torus}} = \mathcal{C}_1 \sum_i \mathcal{Z}_{1i}^3 \prod_{j,k} \mathcal{X}_{ijk} + \mathcal{C}_2 \sum_j \mathcal{Z}_{2j}^3 \prod_{i,k} \mathcal{X}_{ijk} + \mathcal{C}_3 \sum_k \mathcal{Z}_{3k}^3 \prod_{i,j} \mathcal{X}_{ijk}. \quad (4.6)$$

This gauge invariant superpotential W_{torus} now is the source of various F-term constraints. We denote the F-term constraints following from the derivative of W_{torus} with respect to x_{ijk}, z_{ai} and c_a by $F_{x_{ijk}}$, $F_{z_{ai}}$ and F_{c_a}, respectively.

The 27 $F_{x_{ijk}}$ constraints read

$$c_1 z_{1i}^3 \prod_{(\beta,\gamma)\neq(j,k)} x_{i\beta\gamma} + c_2 z_{2j}^3 \prod_{(\alpha,\gamma)\neq(i,k)} x_{\alpha j\gamma} + c_3 z_{3k}^3 \prod_{(\alpha,\beta)\neq(i,j)} x_{\alpha\beta k} = 0 \quad (4.7)$$

and the 9 $F_{z_{ai}}$ terms are

$$c_1 z_{1i}^2 \prod_{\beta,\gamma} x_{i\beta\gamma} = 0, \qquad c_2 z_{2i}^2 \prod_{\alpha,\gamma} x_{\alpha i\gamma} = 0, \qquad c_3 z_{3i}^2 \prod_{\alpha,\beta} x_{\alpha\beta i} = 0. \quad (4.8)$$

From now on, we use the shorthand notation

$$X_{1i} = \prod_{\beta,\gamma} x_{i\beta\gamma}, \qquad X_{2i} = \prod_{\alpha,\gamma} x_{\alpha i\gamma}, \qquad X_{3i} = \prod_{\alpha,\beta} x_{\alpha\beta i}. \quad (4.9)$$

We furthermore find the three torus-generating constraints F_{c_a} to be

$$\begin{aligned} z_{11}^3 X_{11} + z_{12}^3 X_{12} + z_{13}^3 X_{13} &= 0, \\ z_{21}^3 X_{21} + z_{22}^3 X_{22} + z_{23}^3 X_{23} &= 0, \\ z_{31}^3 X_{31} + z_{32}^3 X_{32} + z_{33}^3 X_{33} &= 0. \end{aligned} \quad (4.10)$$

As we have seen in the resolution of $\mathbb{C}^3/\mathbb{Z}_3$, the resolution of $T^6/\mathbb{Z}_3$ now only depends on the value of the FI-parameters a_a and $b_{\alpha\beta\gamma}$ in the

D-term constraints. In general, to resolve the orbifold singularities, we need $a_a, b_{\alpha\beta\gamma} > 0\ \forall a, \alpha, \beta, \gamma$, which is the blow-up phase condition. We will analyze the phase structure a bit more thoroughly in the minimal fully resolvable model, because there we have to deal with less FI-parameters a and b, which makes the discussion much clearer.

4.1.1 Divisors in the maximal fully resolvable model

In [9], three different types of divisors where imposed, which are called ordinary (D_{ai}), exceptional ($E_{\alpha\beta\gamma}$) and inherited divisors (R_a). We now give a proper definition of these divisors in the maximal fully resolvable model.

Ordinary divisors

In [9], the ordinary divisors D_{ai} are defined as divisors which fix the coordinate u_a of the a-th two-torus to the value of an orbifold fixed point. To obtain a proper definition in the Weierstraß coordinates z_{ai}, we take a look back on the orbifold action θ in equation (2.18). Obviously, the fixed points under this action should fulfill the condition $(z_{11}, z_{21}, z_{31}) = (0, 0, 0)$.

One could naively think that the three ordinary divisors in each two-torus are obtained by the three solutions to $z_{a1} = 0$ combined with the corresponding torus defining equation in (4.10), like

$$\frac{z_{12}}{z_{13}} = -\frac{X_{13}}{X_{12}} \cdot \zeta^k \tag{4.11}$$

for the ordinary divisors in the first two-torus. But this is not the case: evidently, these 3 solutions for $k = 1, 2, 3$ are all identified applying the 3-volution action α_1. Hence this can only be one out of three ordinary divisors. This shows clearly that it is indispensable not just to look for fixed points under the action of θ, but to search for fixed points under θ, α_a and the $\mathbb{C}^*$-action of the weighted projective space $\mathbb{P}^6_{1,1,1}$ we are in.

This means we can go the other way around of what we did in (2.13) and write our $\mathbb{Z}_3$ actions again as $z_{ai} \to \zeta \cdot z_{ai}$, $a, i = 1, 2, 3$. We now see clearly that the $3 \cdot 3$ ordinary divisors correspond to

$$D_{ai} := \{z_{ai} = 0\}, \tag{4.12}$$

which is what we use to define the ordinary divisors in the maximal fully resolvable model.

Exceptional divisors

The definition of the exceptional divisors in the GLSM language is easy to give: we simply set to zero the exceptional coordinates $x_{\alpha\beta\gamma}$ which arise in the resolution process,

$$E_{\alpha\beta\gamma} := \{x_{\alpha\beta\gamma} = 0\}\,. \tag{4.13}$$

Inherited divisors

In contrast to the work of [9], we only consider the inherited divisors R_a that fix both the real and imaginary part of the coordinate of only one of the three two-tori, i.e. we do not consider divisors of the type R_{ij}. On the covering space (i.e. the space where we have not divided out the orbifold symmetries yet), we define the inherited divisor R_a by setting the corresponding complex coordinate u_a to a fixed value $\tilde{u}_a$. Since every divisor has to be invariant under the orbifold symmetries, we have to add to the definition the image points of $\tilde{u}_a$ under the 3-volution and orbifold actions. Hence we can write

$$R_a := \bigcup_{k,l=0}^{2} \left\{ u_a = \zeta^k \tilde{u}_a + l \cdot \frac{\zeta - 1}{3} \right\}. \tag{4.14}$$

The most direct way to obtain a hypersurface constraint for R_a in the Weierstraß coordinates z_{ai} we use goes as follows: we map the 9 points in the a-th two-torus that make up the inherited divisor R_a by mapping the first point $\tilde{u}_a$, which we call $(\tilde{z}_{a1}, \tilde{z}_{a2}, \tilde{z}_{a3})$. The 8 remaining points are the image points of $(\tilde{z}_{a1}, \tilde{z}_{a2}, \tilde{z}_{a3})$ under 3-volution and orbifold action, which are given by equation (2.18) and (2.19). Hence we can write the definition of R_a in Weierstraß coordinates as

$$R_a := \left\{ (z_{a1}, z_{a2}, z_{a3}) = \bigcup_{k,l=0}^{2} (\zeta^k z_{a1}, \zeta^{2l} z_{a2}, \zeta^l z_{a3}) \right\}. \tag{4.15}$$

To get the hypersurface equation for R_a, we simply have to think about how it has to look in order to reproduce the 9 points of (4.15) when we compute the intersection set with the equation that defines the a-th two-torus in the projective space $\mathbb{P}^2_{1,1,1} \times \mathbb{P}^2_{1,1,1} \times \mathbb{P}^2_{1,1,1}$ given in the corresponding equation of (4.10).

In order to get a more handy form of our equations where we do not have to care anymore about the $\mathbb{C}^*$ symmetry of $\mathbb{P}^2_{1,1,1}$, we divide (4.10) by z_{a3} and call $\left(\frac{z_{a1}}{z_{a3}}\right) = p_a$ and $\left(\frac{z_{a2}}{z_{a3}}\right) = q_a$ to get

$$X_{a1} p_a^3 + X_{a2} q_a^3 = -X_{a3}. \tag{4.16}$$

We see that the orbifold action θ and the 3-volution α_a act as

$$\begin{aligned} \theta: &\qquad p_a \to \zeta p_a, \qquad q_a \to q_a, \\ \alpha_a: &\qquad p_a \to \zeta^2 p_a, \qquad q_a \to \zeta q_a, \end{aligned} \tag{4.17}$$

on our new coordinates p_a, q_a.

Since θ and α_a form a basis of $\mathbb{Z}_3$ actions on p_a and q_a, a hypersurface equation for R_a in $\begin{pmatrix} p_a \\ q_a \end{pmatrix}$ coordinates has to reproduce all the 9 points

$$\begin{pmatrix} p_a \\ q_a \end{pmatrix} = \bigcup_{k,l=0}^{2} \left\{ \begin{pmatrix} \zeta^k \tilde{p}_a \\ \zeta^l \tilde{q}_a \end{pmatrix} \right\} \tag{4.18}$$

in the intersection set with equation (4.16), where $\tilde{p}_a = \frac{\tilde{z}_{a1}}{\tilde{z}_{a3}}$, $\tilde{q}_a = \frac{\tilde{z}_{a2}}{\tilde{z}_{a3}}$.

The only possible equation for R_a reproducing such a solution set is a third degree polynomial in p_a and q_a,

$$R_a := \left\{ p_a^3 + A_a q_a^3 = B_a \right\}, \tag{4.19}$$

with some complex constants A_a and B_a depending on $\begin{pmatrix} \tilde{p}_a \\ \tilde{q}_a \end{pmatrix}$.

Multiplying this equation with z_{a3}^3, we get back our Weierstraß coordinates of the weighted projective space $\mathbb{P}^2_{1,1,1}$, where R_a reads in the maximal fully resolvable model

$$a_{a1}(\tilde{u}_a) z_{a1}^3 + a_{a2}(\tilde{u}_a) z_{a2}^3 + a_{a3}(\tilde{u}_a) z_{a3}^3 = 0, \tag{4.20}$$

with some complex coefficients a_{ai} depending on the place $\tilde{u}_a$ of the inherited divisor on the covering space.

Since we want the inherited divisors R_a to be fully gauge invariant, we have to multiply the coordinates z_{ai} in (4.20) with the exceptional coordinates $x_{\alpha\beta\gamma}$ on the right places. The final defining equations for the inherited divisors in the maximal fully resolvable model then read

$$R_a := \left\{ a_{a1} X_{a1} z_{a1}^3 + a_{a2} X_{a2} z_{a2}^3 + a_{a3} X_{a3} z_{a3}^3 = 0 \right\}. \tag{4.21}$$

as it was already found in [7].

Equivalence relations

From (4.21) we can read off an equivalence relation between the inherited, ordinary and exceptional divisors as it was done in [7]. They read

$$R_1 \sim 3D_{11} + \sum_{\beta,\gamma} E_{1\beta\gamma} \sim 3D_{12} + \sum_{\beta,\gamma} E_{2\beta\gamma} \sim 3D_{13} + \sum_{\beta,\gamma} E_{3\beta\gamma}, \tag{4.22}$$

and analogously for R_2, R_3.

4.1.2 Intersection numbers

In this section, we will compute the triple intersection numbers (i.e. the number of distinct intersection points of three divisors) of the divisors we just introduced in both the orbifold and the resolved blow-up phase.

The orbifold phase

The orbifold phase is characterized by $b_{\alpha\beta\gamma} < 0 < a_a$. We directly see in the D-term constraint (4.5) that the exceptional coordinates $x_{\alpha\beta\gamma}$ are forced to have a VEV, hence there are no exceptional Divisors $E_{\alpha\beta\gamma} := \{x_{\alpha\beta\gamma} = 0\}$ possible. This means we just have to calculate the triple intersection numbers involving only ordinary and inherited divisors.

In principle, we can obtain all the intersection numbers of the orbifold phase without using all the divisor-defining equations we derived previously, we only have to think about how many points the divisors fix on the cover. As already stated, the inherited divisors are defined by fixing 9 points on the cover, one point and its 8 images under 3-volution and orbifold action. Contrary, since the ordinary divisors fix one coordinate to the value of an orbifold fixed point, they are only defined by one point (the images under orbifold action and 3-volution would again be that point). Hence it is easy to derive the intersection numbers by simply multiplying the numbers of points the divisors define. After that, we only have to divide out all the orbifold symmetries by only counting once solutions that can be identified using combinations of θ and α_a. We directly get

$$D_{1i}D_{2j}D_{3k} = 1, \quad D_{1i}D_{2j}R_3 = 1, \quad D_{1i}R_2R_3 = 3, \quad R_1R_2R_3 = 9, \quad (4.23)$$

for $i, j, k = 1, 2, 3$.

We can also determine these intersection numbers in a GLSM way using the F- and D-term constraints as well as the divisor hypersurface equations we derived. In the resolved blow-up phase, this will be the only consistent way to go.

In the orbifold phase, since all $x_{\alpha\beta\gamma}$ have a VEV and since at least one z_{ai} for each index a has to be nonzero (see eq. (4.4), $a_a > 0$), the $F_{z_{ai}}$-constraints (4.8) force all c_a to vanish. The only nontrivial F-term constraints are then F_{c_a}, (4.10).

We can compute $D_{1i}D_{2j}D_{3k}$ plugging in the definition $D_{al} := \{z_{al} = 0\}$ into the F_{c_a} constraints (4.10) and counting the number of solutions. Clearly, we get 3 cubic equations which give 3 solutions differing by a factor of ζ^n, $n = 0, 1, 2$ each. In total, these are $3 \cdot 3 \cdot 3$ solutions which all can be identified using the 3-volutions α_a and the orbifold symmetry θ. Hence this confirms that $D_{1i}D_{2j}D_{3k} = 1$.

Next, $D_{1i}D_{2j}R_3$ can be computed inserting the definitions of D_{1i} and D_{2j} in the first two equations of (4.10). Furthermore, we have a homogeneous third degree polynomial for R_3 as the third one in (4.21) and the last equation of (4.10) left over. Hence we have four homogeneous degree three polynomials which we can intersect to obtain 3^4 solutions differing again by factors of ζ^n. Once more, these 3^4 solutions all can be identified using the θ and α_a actions. So as expected, we get $D_{1i}D_{2j}R_3 = 1$.

For $D_{1i}R_2R_3$ we end up with five homogeneous cubic polynomials, giving 3^5 solutions which differ by factors of powers of ζ again. This time, since we just have four independent $\mathbb{Z}_3$ actions at our disposal, we only can identify the solutions in groups of 3^4 each, which means that we obtain $D_{1i}R_2R_3 = 3$, as we should.

Finally, for $R_1R_2R_3$, we have three cubic equations for all the R_a and three cubic polynomials out of the F_{c_a}-term constraints. So with six homogeneous cubic equations we naturally get 3^6 solutions that we can identify in groups of 3^4 using θ and α_a to obtain $R_1R_2R_3 = \frac{3^6}{3^4} = 9$.

All these triple intersections are consistent with the D-term constraints and the phase condition $b_{\alpha\beta\gamma} < 0 < a_a$, as the reader can easily check himself.

The blow-up phase

The blow-up phase is defined by $0 < b_{\alpha\beta\gamma} << a_a$. Since $b_{\alpha\beta\gamma} > 0$, the $x_{\alpha\beta\gamma}$ do not necessarily have to have a VEV, so in this phase, exceptional divisors $E_{\alpha\beta\gamma} := \{x_{\alpha\beta\gamma} = 0\}$ occur. This is the phase where the orbifold singularities are resolved, thus we should get the same results as in [9], which are presented in chapter 3.

Clearly, in the orbifold phase, R_a and D_{ai} did not intersect by definition. With the D- and F-term constraints as well as the hypersurface equations for the divisors at hand, we can show that this remains the same in the blow-up phase. We insert the D_{ai} condition $z_{ai} = 0$ in the corresponding F-term constraint (4.10) and the defining equation for R_a to obtain two linear independent homogeneous polynomials in z_{aj}^3 and z_{ak}^3 for $j, k \neq i$. We can combine these polynomials to get the conditions

$$X_{aj} \cdot z_{aj}^3 = X_{ak} \cdot z_{ak}^3 = 0, \tag{4.24}$$

If one of the z-coordinates is zero, then atomatically by (4.10) all the three coordinates z_{ai}, $i = 1, 2, 3$ are zero, which violates the blow-up phase condition $a_a > 0$ in (4.4). Hence R_a and D_{ai} do not intersect. There is a complete analogous way to show that R_a does not intersect with E_{ijk}. This was done explicitly in [7], so we will not do it again here. This directly means taking use of (4.22) that R_a does not self-intersect.

Next, let us investigate $R_1R_2R_3$. Since R_a does not intersect with E_{ijk} we surely know that on the divisors R_a, all exceptional coordinates $x_{\alpha\beta\gamma}$ have to have a VEV. Thus, with the definitions of the inherited divisors (4.21) and the F_{c_a}-term constraints (4.10), we have six homogeneous polynomials of degree three of which we want to have the intersection set. As in the orbifold phase, these six cubic equations give 3^6 solutions that differ by factors of potentials of ζ. Hence these solutions can again be identified in groups of 3^4 by the orbifold action θ and the three 3-volutions α_a we have in that model. Thus we get $R_1R_2R_3 = \frac{3^6}{3^4} = 9$ as we also had in the orbifold phase.

As we know that $R_a \cap E_{ijk} = \{\ \}$, we can take the equivalence relations (4.22) to determine the intersection numbers involving R_a and $D_{a'i}$, with $a' \neq a$. We compute

$$R_1R_2D_{3i} = R_1R_2\frac{1}{3}\left(R_3 - \sum_{j,k} E_{jki}\right) = \frac{1}{3}R_1R_2R_3 = 3. \tag{4.25}$$

Analogously, we get

$$R_1D_{2i}D_{3j} = R_1\frac{1}{3}\left(R_2 - \sum_{k,l} E_{kil}\right)\frac{1}{3}\left(R_3 - \sum_{k,l} E_{klj}\right) = \frac{1}{9}R_1R_2R_3 = 1. \tag{4.26}$$

The only intersection that vanishes in the blow-up but was present in the orbifold phase is the intersection $D_{1i}D_{2j}D_{3k}$: We directly see that $z_{1i} = z_{2j} = z_{3k} = 0$ would contradict the blow-up phase condition $0 < b_{\alpha\beta\gamma} << a_a$, because $b_{ijk} = -3|x_{\alpha\beta\gamma}|^2 < 0$. It is nice to see that the orbifold fixed point $D_{1i}D_{2j}D_{3k}$ is present in the orbifold phase, but vanishes in the blow-up phase. This is where the GLSM resolution process delivers a very intuitive picture of how we get rid of orbifold singularities using these methods.

Next we consider intersections containing at least two different exceptional divisors, E_{ijk} and $E_{i'j'k'}$, with $(i,j,k) \neq (i',j',k')$. Looking at (4.10), we see that an exceptional divisor E_{ijk} sets one term to zero in each of the three torus defining equations of (4.10). A further exceptional divisor $E_{i'j'k'}$ would set to zero at least one other term in the three equations (4.10), so in at least one of these three equations, we get something like

$$X_{ai}z_{ai}^3 = 0. \tag{4.27}$$

This always indicates a violation of the blow-up phase condition since in this case we can always construct relations like

$$\sum_{\alpha\beta\gamma} k_{\alpha\beta\gamma}b_{\alpha\beta\gamma} > \sum_a k_a a_a, \tag{4.28}$$

with some coefficients $k_{\alpha\beta\gamma}, k_a$ which are either 0 or 1. This contradicts the condition $0 < b_{\alpha\beta\gamma} << a_a$, hence we conclude that there are no intersections of two or more different exceptional divisors in the blow-up phase.

Since we know that different exceptional divisors do not intersect, we can use the equivalence relations (4.22) to compute the triple self-intersection E_{ijk}^3:

$$E_{ijk}^3 = (R_1 - 3D_{1i})(R_2 - 3D_{2j})(R_3 - 3D_{3k}) = 9. \tag{4.29}$$

Superfield U(1) charges\ s. comp.	$\mathcal{Z}_{1i}$ z_{1i}	$\mathcal{Z}_{2j}$ z_{2j}	$\mathcal{Z}_{3k}$ z_{3k}	$\mathcal{C}_1$ c_1	$\mathcal{C}_2$ c_2	$\mathcal{C}_3$ c_3	$\mathcal{X}_{111}$ x_{111}
R_1	1	0	0	-3	0	0	0
R_2	0	1	0	0	-3	0	0
R_3	0	0	1	0	0	-3	0
E_{111}	δ_{i1}	δ_{j1}	δ_{k1}	0	0	0	-3

Table 4.3: Charge assignment for $T^6/\mathbb{Z}_3$ in the minimal fully resolvable model

Furthermore, using (4.22), one gets

$$D_{1i}D_{2j}E_{ijk} = \frac{1}{9}\sum_{i,j} E_{ijk}^3 = 1, \qquad D_{ai}E_{ijk}^2 = -\frac{1}{3}E_{ijk}^3 = -3. \tag{4.30}$$

4.2 The minimal fully resolvable model

In the minimal fully resolvable model, which appeared first in [11], only one exceptional gauging is needed to resolve all the 27 orbifold singularities of $T^6/\mathbb{Z}_3$ at once. The charge assignment is given in table 4.3, where x_{111} is the exceptional coordinate corresponding to the $U(1)_{E_{111}}$ gauging.

This directly leads to the D-term constraints

$$\begin{aligned} |z_{a1}|^2 + |z_{a2}|^2 + |z_{a3}|^2 - 3|c_a|^2 &= a_a, \\ |z_{11}|^2 + |z_{21}|^2 + |z_{31}|^2 - 3|x_{111}|^2 &= b_{111}. \end{aligned} \tag{4.31}$$

The gauge invariant torus producing superpotential W_{torus} looks like

$$W_{\text{torus}} = \sum_a \mathcal{C}_a \left(z_{a1}^3 x_{111} + z_{a2}^3 + z_{a3}^3\right), \tag{4.32}$$

hence we obtain the F-term constraint $F_{x_{111}}$

$$c_1 z_{11}^3 + c_2 z_{21}^3 + c_3 z_{31}^3 = 0, \tag{4.33}$$

the $F_{z_{ai}}$ terms are

$$c_a z_{a1}^2 x_{111} = 0, \qquad c_a z_{a2}^2 = 0, \qquad c_a z_{a3}^2 = 0, \tag{4.34}$$

and the torus-defining constraints F_{c_a} read

$$x_{111} z_{a1}^3 + z_{a2}^3 + z_{a3}^3 = 0. \tag{4.35}$$

4.2.1 Divisors in the minimal fully resolvable model

According to equation (4.3), the exceptional gauging $U(1)_{E_{111}}$ spares only one $\mathbb{Z}_3$ action on the Weierstraß coordinates z_{ai}, rather than the four actions $\mathbb{Z}_{3,\text{orbi}} \times \mathbb{Z}^3_{3,3\text{-vol}}$ we had in the maximal fully resolvable model. This action is the orbifold action θ in (2.18). Hence all divisors in the minimal fully resolvable model have to be invariant under the orbifold action θ, but not anymore under the 3-volutions α_a. This leads to some necessairy modifications in their respective hypersurface equations. We will work out the proper definitions in the following.

Ordinary divisors

An ordinary divisor fixes two real coordinates of the orbifold to the value of an orbifold fixed point [9]. Since the only orbifold symmetry in the minimal fully resolvable model is

$$\theta : \ (z_{11}, z_{21}, z_{31}) \to (\zeta z_{11}, \zeta z_{21}, \zeta z_{31}),$$

clearly, an orbifold fixed point must fulfill the condition $(z_{11}, z_{21}, z_{31}) = (0, 0, 0)$. Hence we obtain the ordinary divisors in the minimal fully resolvable model by setting one of the coordinates z_{a1} to zero. Naively, one could have the impression that there is just one orbifold fixed point and only the three corresponding ordinary divisors D_{a1} exist. But this is not quite right: combining the condition $z_{a1} = 0$ with the F_{c_a}-term constraints (4.35), we obtain three distinct points for each divisor D_{a1}, namely the three solutions to

$$\left(\frac{z_{a2}}{z_{a3}}\right)^3 = -1, \qquad \frac{z_{a2}}{z_{a3}} = -\zeta^{\tilde{n}}, \qquad \tilde{n} = 0, 1, 2. \tag{4.36}$$

This means that an ordinary divisor that is equivalent to an ordinary divisor in the maximal fully resolvable model has to be defined by

$$D_{an} := \left\{ z_{a1} = 0 \;\middle|\; \frac{z_{a2}}{z_{a3}} = -\zeta^n \right\}. \tag{4.37}$$

In particular, we have to keep in mind that the number of our solutions of the equation system with the torus defining equations (4.35) and the hypersurface constraints $z_{a1} = 0$ will be raised by a factor of 3 for each ordinary divisor involved in the triple intersection number we compute. The reason for this is that by setting $z_{a1} = 0$, we actually compute the intersection with $\bigcup_{n=0}^{2} D_{an}$ rather than with D_{an}.

Furthermore, we will see that in the minimal fully resolvable model, the conditions $z_{a2} = 0$ and $z_{a3} = 0$ do *not* define an ordinary, but an inherited divisor!

Exceptional divisors

Since the orbifold symmetries never affect the exceptional coordinates, the definition of an exceptional divisor is in every model simply given by setting the associated exceptional coordinate to zero. Hence in the minimal fully resolvable model, we only have one exceptional divisor,

$$E_{111} := \{x_{111} = 0\}\,. \tag{4.38}$$

Inherited divisors

To find a proper hypersurface equation for the inherited divisors R_a in the minimal fully resolvable model, we proceed analogously as we did in the maximal fully resolvable model.

On the covering space, the inherited divisor R_a is again defined by setting the coordinate u_a to some fixed value $u_a = \tilde{u}_a$ nonequal to an orbifold fixed point. Once more, we want our inherited divisor to be invariant under all the symmetries our resolution model possesses. Since there is just the orbifold action θ in this model, the inherited divisor is now defined by only three points on the cover rather than nine as it was in the maximal model. These three points are the point $u_a = \tilde{u}_a$ and its two image points under θ (2.18), hence R_a is defined by

$$R_a := \left\{ u_a = \bigcup_{k=0}^{2} \zeta^k \tilde{u}_a \right\}. \tag{4.39}$$

To transform this definition to the Weierstraß coordinates z_{ai}, we simply call $(\tilde{z}_{a1}, \tilde{z}_{a2}, \tilde{z}_{a3})$ the mapped point of $\tilde{u}_a$ and add the image points under the orbifold action θ as it acts in these coordinates. We then obtain

$$R_a := \left\{ (z_{a1}, z_{a2}, z_{a3}) = \bigcup_{k=0}^{2} (\zeta^k \tilde{z}_{a1}, \tilde{z}_{a2}, \tilde{z}_{a3}) \right\}. \tag{4.40}$$

To get a hypersurface equation for R_a in the projective space of our Weierstraß coordinates, we simply have to think about how such an equation has to look in order to reproduce the three points that define R_a in (4.40) when we intersect it with the corresponding F_{c_a}-term constraint in (4.35).

Since (4.35) is cubic in all z_{ai}, we necessarily need a linear equation for R_a in order to get three solution points in the intersection set of both equations. To see this, we use the coordinates $p_a = \frac{z_{a2}}{z_{a1}}$, $q_a = \frac{z_{a3}}{z_{a1}}$ to get rid of the $\mathbb{C}^*$-symmetry. Then (4.35) becomes

$$p_a^3 + q_a^3 = -x_{111}. \tag{4.41}$$

Clearly, the orbifold action θ acts as

$$\theta: \qquad p_a \to \zeta p_a, \qquad q_a \to \zeta q_a \tag{4.42}$$

on the coordinates p_a, q_a. Hence, we need to find an equation that has the intersection set

$$\begin{pmatrix} p_a \\ q_a \end{pmatrix} = \bigcup_{k=0}^{2} \left\{ \zeta^k \begin{pmatrix} \tilde{p}_a \\ \tilde{q}_a \end{pmatrix} \right\} \tag{4.43}$$

with equation (4.41), where $\tilde{p}_a = \frac{\tilde{z}_{a2}}{\tilde{z}_{a1}}$, $\tilde{q}_a = \frac{\tilde{z}_{a3}}{\tilde{z}_{a1}}$. Such an equation necessarily has the form

$$a_{a2} p_a + a_{a3} q_a = 0, \tag{4.44}$$

with some complex constants a_{a2}, a_{a3} depending on $\begin{pmatrix} \tilde{p}_a(\tilde{u}_a) \\ \tilde{q}_a(\tilde{u}_a) \end{pmatrix}$. We multiply this equation by z_{a1} to get the hypersurface constraint for R_a in Weierstraß coordinates,

$$R_a := \{ a_{a2}(\tilde{u}_a) z_{a2} + a_{a3}(\tilde{u}_a) z_{a3} = 0 \} . \tag{4.45}$$

We see that, as already mentioned, the constraints $z_{a2} = 0$ and $z_{a3} = 0$ indeed correspond to an inherited divisor R_a, namely the one where either $a_{a3}(\tilde{u}_a) = 0$ or $a_{a2}(\tilde{u}_a) = 0$, respectively. An inherited divisor with $a_{a2} = a_{a3}$ forces $z_{a1} = 0$ in the orbifold phase and hence corresponds to $\bigcup_{n=0}^{2} D_{an}$.

There is another way to see that the inherited divisors in the minimal fully resolvable model have to be homogeneous linear polynomials. It is analogous to the way used in [7] to show that the inherited divisors in the maximal fully resolvable model are given by cubic polynomials.

It goes as follows: from the Weierstraß mapping (2.9) we see that

$$y_a = p_a v_a \tag{4.46}$$

with $p_a = \epsilon_1^{-3/2} \frac{\wp'(\tilde{u}_a)}{2}$ for some point $\tilde{u}_a$. Since this equation is invariant under $\mathbb{Z}_{3,orbi}$, it is valid for all the three points within the union of (4.39), so it can be seen as the hypersurface constraint for R_a. We take the relations

$$y_a = \frac{\sqrt{3}}{2^{2/3}} (z_{a2} - z_{a3}), \qquad v_a = 2^{-2/3} (z_{a2} + z_{a3}) \tag{4.47}$$

to write (4.46) in the form

$$(p_a(\tilde{u}_a) - 1) z_{a2} + (p_a(\tilde{u}_a) + 1) z_3 = a_{a2}(\tilde{u}_a) z_{a2} + a_{a3}(\tilde{u}_a) z_{a3} = 0, \tag{4.48}$$

which exactly reproduces the equation found in (4.45).

Equivalence relations

In order to get an equivalence relation between D_{ai}, R_a and E, we combine the definition (4.45) of R_a with the always valid torus-equation (4.35) to get

$$x_{111} z_{a1}^3 + \left(1 - \left(\frac{a_{a2}}{a_{a3}} \right)^3 \right) z_{a2}^3 = 0. \tag{4.49}$$

From the first term we read off the equivalence relation

$$R_a \sim 3D_{ai} + E_{111}. \tag{4.50}$$

The root of the second term would set $z_{a2} = z_{a3} = 0$ and thus would lead to the same equivalence relations after a short calculation.

4.2.2 Intersection numbers in various phases of the minimal fully resolvable model

In [7], a thorough analysis of the various phases of the minimal fully resolvable model was given, so we will not redo that here. What was not done before is the computation of the intersection numbers in any of these phases of the minimal fully resolvable model, so we will focus on that in this section.

To simplify our life in the following, we set the FI-parameters a_a all to the same value, $a_a = a$. Furthermore, we use a more compact notation and call $b_{111} = b$ and $x_{111} = x$. The whole set of D- and F-term constraints then reads

$$|z_{a1}|^2 + |z_{a2}|^2 + |z_{a3}|^2 - 3|c_a|^2 = a, \tag{4.51}$$
$$|z_{11}|^2 + |z_{21}|^2 + |z_{31}|^2 - 3|x|^2 = b, \tag{4.52}$$
$$c_1 z_{11}^3 + c_2 z_{21}^3 + c_3 z_{31}^3 = 0, \tag{4.53}$$
$$c_a z_{a1}^2 x = c_a z_{a2}^2 = c_a z_{a3}^2 = 0, \tag{4.54}$$
$$x z_{a1}^3 + z_{a2}^3 + z_{a3}^3 = 0. \tag{4.55}$$

So effectively, we only have the two parameters a, b that determine the whole phase structure of our resolution model.

A preliminary remark on the inherited divisors R_a

With (4.45) and (4.55), we can replace the modulus of the coordinate $|z_{a1}|$ by

$$|z_{a1}| = \left| \sqrt[3]{\frac{(a_{a2}/a_{a3})^3 - 1}{x}} \right| \cdot |z_{a2}| = \left| \sqrt[3]{\frac{(a_{a3}/a_{a2})^3 - 1}{x}} \right| \cdot |z_{a3}|. \tag{4.56}$$

Hence we can write equation (4.51) as

$$\begin{aligned} &\left(1 + \left|\frac{x}{(a_{a2}/a_{a3})^3 - 1}\right|^{2/3} + \left|\frac{x}{(a_{a3}/a_{a2})^3 - 1}\right|^{2/3}\right) |z_{a1}|^2 - 3|c_a|^2 = \\ &= F(a_{a2}, a_{a3}, x)|z_{a1}|^2 - 3|c_a|^2 = a. \end{aligned} \tag{4.57}$$

So, for the intersection of $R_1 R_2 R_3$ we get a constraint like

$$\sum_a \frac{a + 3|c_a|^2}{F(a_{a2}, a_{a3}, x)} - 3|x|^2 = b. \tag{4.58}$$

Phase	Dimension	D_{ai}	R_a	E
$a, b < 0$	0	triv.	nonex.	nonex.
$b < 0 < a$	3	ex.	ex.	nonex.
$0 < b < a$	3	ex.	ex.	ex.
$0 < a < b < 2a$	3	ex.	ex.	ex.
$0 < 2a < b < 3a, c_1, c_2 \neq 0, c_3 = 0$	1	D_{3n_3}	R_3	triv.
$0 < 2a < b < 3a, c_a = 0\ \forall a$	3	ex.	ex.	ex.
$0 < 3a < b, c_a \neq 0\ \forall a$	1	nonex.	triv.	triv.
$0 < 3a < b, c_1, c_2 \neq 0, c_3 = 0$	1	D_{3n_3}	R_3	triv.

Table 4.4: Resolution phases in the minimal fully resolvable model and their target space dimension. The existence of the divisors D_{ai}, R_a and E is specified, whereas triv. means that the corresponding hypersurface constraint is fulfilled trivially. In two phases, only the divisors D_{3n_3} and R_3 are nontrivial.

Hence, for given ranges of a and b in a specific phase, it depends on the value of the exceptional coordinate x and on the location of the inherited divisors R_a encoded in a_{a2} and a_{a3} if there is an intersection or a violation of the phase condition. The latter would mean that no intersection is possible, which is a rather surprising fact. One can easily generalize the constraint (4.58) to intersections involving both inherited and ordinary divisors: for an ordinary divisor $D_{a'n_i}$, one just has to leave out the a'-term in the sum over a.

In our computation of intersection numbers in the following, we will only consider divisors that fulfill this condition, so we do not have to care about it anymore.

$a, b < 0$: The non-geometric regime

In this phase, all the c_a and x have to be nonzero. Hence all z_{ai} are forced to be zero, $z_{ai} = 0\ \forall a, i = 1, 2, 3$. So the target space collapses to a single point. Exceptional and inherited divisors are obviously nonexistent in this phase, whereas the ordinary divisors D_{an} are part of the target space geometry.

$b < 0 < a$: The orbifold phase

Since b has to be negative, we see on the D-term constraint (4.52) that x has to have a VEV, hence no exceptional divisors are possible.

For $R_1 R_2 R_3$, we combine the R_a hypersurface equations (4.45) with the F_{c_a}-term constraints (4.55) to obtain $3 \cdot 3 \cdot 3$ solution points that differ by factors of potentials of ζ on the z_{a1} coordinates. Hence we can identify these solutions in groups of three using the orbifold action θ. The intersection number $R_1R_2R_3$ then reduces to $R_1R_2R_3 = \frac{3\cdot3\cdot3}{3} = 9$, as we have already found in the orbifold phase of the maximal fully resolvable model.

Computing $R_1R_2D_{3n}$, we use the hypersurface constraint $z_{31} = 0$ for D_{3n}, keeping in mind that we thereby actually compute $R_1R_2\left(\bigcup_{n=0}^{2} D_{3n}\right)$. Thus, we have to divide our solution by a factor of 3 afterwards. Intersecting the hypersurface equations of the divisors with the torus defining constraints (4.55), we get $3\cdot3\cdot3 = 27$ solutions. Again, we can identify these solutions in sets of 3 taking use of the orbifold action θ. Hence, $R_1R_2D_{3n} = \frac{1}{3}\cdot\frac{3\cdot3\cdot3}{3} = 3$, as we have expected from the maximal fully resolvable model. Analogously, we get $R_1D_{2n_2}D_{3n_3} = 1$ and $D_{1n_1}D_{2n_2}D_{3n_3} = 1$ as we should.

$0 < b < a$: Blow-up phase I

Here, the D-term constraints forbid the sets of coordinates $\{z_{a1}, z_{a2}, z_{a3}\}$, $\{z_{11}, z_{21}, z_{31}\}$ and $\{x, z_{a2}, z_{a3}\}$ to vanish. This directly implies that $c_a = 0$ $\forall a$.

Since $(z_{11}, z_{21}, z_{31}) \neq (0,0,0)$, all the orbifold fixed points have disappeared, hence this phase is fully resolved. This includes that $D_{1n_1}D_{2n_2}D_{3n_3} = 0$.

Again we see that R_a does never intersect with D_{an}, because this would set $\{z_{a1}, z_{a2}, z_{a3}\}$ to zero, which is not allowed in this phase. Furthermore, an intersection of E with R_a would lead to $b > a$, hence they do not intersect either.

Assuming that the following inherited divisors are chosen such that they fulfill (4.58), we again get

$$R_1R_2R_3 = 9, \qquad R_1R_2D_{3n_3} = 3, \qquad R_1D_{2n_2}D_{3n_3} = 1. \tag{4.59}$$

We can make use of the equivalence relation (4.50) to determine the intersection numbers involving E,

$$D_{1i}D_{2j}E = 1, \qquad D_{ai}E^2 = -3, \qquad E^3 = 9. \tag{4.60}$$

As already shown in [7], the blow-up phase I has a sharp upper bound on the value of b. Considering the intersection curve of two ordinary divisors, let us take D_{1n_1} and D_{2n_2}, we get

$$a - b = |z_{32}|^2 + |z_{33}|^2 + 3|x|^2. \tag{4.61}$$

We see that this is impossible for $b > a$. Hence $b = a$ marks the limit where on the one side ($b < a$) the curve $D_{an_i}D_{a'n_i'}$, $a \neq a'$ exists whereas it vanishes on the other side ($b > a$). This leads us to the next phase.

$0 < a < b < 2a$: Blow-up phase II

The first thing we check is if the coordinates c_a vanish again. If we consider that only one c_a, for example c_1, is nonzero, we immediately see that (4.53) forces z_{11} and (4.54) forces z_{12} and z_{13} to be zero. Hence a would be negative, which is not allowed. Also, it is impossible that all the three c_a are nonzero for $b < 2a$: if they would, we could write $b = 3a + \sum_a |c_a|^2$ which would mean that $b > 3a$. Furthermore, for only two of the three c_a being nonzero (e.g. c_1 and c_2), we would get $b = 2a + 3|c_1|^2 + 3|c_2|^2 + |z_{31}|^2 > 2a$, which is the reason why we set the upper bound of the present phase to $b \nearrow 2a$.

As we have found out a few lines above, intersection numbers with more than one ordinary divisor are forced to vanish in this phase.

For inherited divisors satisfying (4.58), we find the nonvanishing intersection numbers

$$R_1 R_2 R_3 = 9, \qquad R_1 R_2 D_{3n} = 3. \tag{4.62}$$

To obtain the intersection numbers involving the exceptional divisor E, we use the equivalence relation (4.50) to see that

$$R_1 R_2 E = 0, \qquad R_1 D_{2i} E = 3, \qquad D_{ai} E^2 = -3, \qquad R_1 E^2 = 0, \qquad E^3 = 9. \tag{4.63}$$

$0 < 2a < b < 3a$: The critical blow-up phase

This phase branches in two components that differ by their dimensionality. As we have seen in the discussion of the last phase, either all the c_a are zero, or two of them are nonzero. We start with the latter case.

The case c_1 and $c_2 \neq 0$, $c_3 = 0$:

In this case, $z_{12} = z_{13} = z_{22} = z_{23} = 0$. The remaining F-term constraints are

$$z_{32}^3 + z_{33}^3 = 0, \qquad c_1 z_{11}^3 + c_2 z_{21}^3 = 0. \tag{4.64}$$

Hence, the target space dimension becomes $d = 9-4-2-1-1 = 1$, so triple intersection numbers do not make any sense anymore here. Already a single divisor intersects with the target space to give distinct solution points.

Because of $z_{12} = z_{13} = z_{22} = z_{23} = 0$, $a = |z_{11}|^2 - 3|c_1|^2 = |z_{21}|^2 - 3|c_2|^2$. Thus, the ordinary divisors D_{1n_1} and D_{2n_2} do not exist here. Besides, $x = 0$ has to be always fulfilled so E does not give any further constraint.

Furthermore, the hypersurface equations for R_1 and R_2 (4.45) are trivially fulfilled since $z_{a2} = z_{a3} = 0$. So the only existing divisors in this phase are D_{3n_3} and R_3.

The R_3-defining equation $a_{32} z_{32} + a_{33} z_{33} = 0$ intersected with the first equation in (4.64) leads to $z_{32} = z_{33} = 0$. Hence the only nonvanishing coordinates are z_{11}, z_{21}, z_{31}. Since we just have one single nonvanishing coordinate in each two-torus, their moduli are fixed by the D-term constraints to a nonzero value. Their $U(1)$ phases are physically identified, so we conclude that there is only one intersection point of R_3 with the target space.

For D_{3n_3} we set $z_{31} = 0$ so that $z_{11}, z_{21}, z_{32}, z_{33}$ are the only nonzero coordinates. As argued above, the values of z_{11} and z_{21} are fixed by the D-term constraints, so we only have z_{32}, z_{33} and the first equation in (4.64) left over. The intersection number of $\bigcup_{k=1}^{3} D_{3k}$ is then given by the number of solutions to $\frac{z_{32}^3}{z_{33}^3} = -1$, which is 3. So the intersection number of one specific D_{3n_3} with the target space is also 1.

The case $c_a = 0\ \forall a = 1,2,3$ **:**

In this component, we get back the D- and F-term constraints we already know from the orbifold- and the blow-up phases I+II,

$$
\begin{aligned}
&|z_{a1}|^2 + |z_{a2}|^2 + |z_{a3}|^2 = a, \qquad |z_{11}|^2 + |z_{21}|^2 + |z_{31}|^2 - 3|x|^2 = b,\\
&\qquad x z_{a1}^3 + z_{a2}^3 + z_{a3}^3 = 0,
\end{aligned}
$$

hence we also have the same target space dimension, d = 3. Since $b > a$, it is impossible to have intersections with two ordinary divisors, as we already argued in blow-up I+II. Thus, we get all the same intersection numbers as in blow-up phase II.

If we sum up the three D-term equations for a and substract the one for b, we get

$$
3a - b = \sum_a \left(|z_{a2}|^2 + |z_{a3}|^2\right) + 3|x|^2. \tag{4.65}
$$

This means that b can not grow bigger than $3a$, so $b = 3a$ marks the upper bound of this phase.

$0 < 3a < b$**: The over-blow-up phase**

In the over-blow-up regime, at least two c_a are forced to be nonzero to avoid phase contradictions. This directly entails that since $a > 0$, the exceptional coordinate x has to vanish. Hence the exceptional divisor E does not establish a further constraint.

The F- and D-term constraints are truncated to

$$
\begin{aligned}
&c_1 z_{11}^3 + c_2 z_{21}^3 + c_3 z_{31}^3 = 0, \qquad c_a z_{a2}^2 = c_a z_{a3}^2 = 0, \qquad z_{a2}^3 + z_{a3}^3 = 0, \qquad (4.66)\\
&|z_{a1}|^2 + |z_{a2}|^2 + |z_{a3}|^2 - 3|c_a|^2 = a, \qquad |z_{11}|^2 + |z_{21}|^2 + |z_{31}|^2 = b.
\end{aligned}
$$

This phase also splits in two branches:

The case where all $c_a \neq 0$:

In the second equation in (4.66), it is easy to see that in this component, $z_{a2} = z_{a3} = 0$, $a = 1, 2, 3$. Thus,

$$|z_{a1}|^2 - 3|c_a|^2 = a. \tag{4.67}$$

Since $a > 0$, z_{a1} is not allowed to be zero, which means that there are no ordinary divisors present. Furthermore, the hypersurface constraint for the inherited divisors $R_a := \{a_{a2}z_{a2} + a_{a3}z_{a3} = 0\}$ is trivially fulfilled. Hence we can not give any intersection number here.

The case where $c_b = 0$, $c_{a\neq b} \neq 0$:

Assuming that $c_1, c_2 \neq 0$, $c_3 = 0$, this phase leads to exact the same constraints as in the $c_1, c_2 \neq 0$, $c_3 = 0$ case of the critical blow-up regime, equation (4.64). Again, the only existing divisors in the one-dimensional target space are R_3 and D_{3n_3}. For values of z_{ai} that are in accord with the D-term constraints, we infer the same intersection numbers as in the corresponding component of the critical blow up phase, $R_3 = 1$ and $D_{3n_3} = 1$.

One could say that the phase transition between critical blow-up and over-blow-up does not happen in the $c_b = 0$, $c_{a\neq b} \neq 0$ component, but occurs between the two other components.

4.3 A fully resolvable model with the exceptional coordinates x_{111}, x_{211} and x_{311}

In this section, we discuss a further fully resolvable model with three exceptional gaugings $U(1)_{E_{111}}, U(1)_{E_{211}}, U(1)_{E_{311}}$. We take an analogous charge assignment as in the last two sections which can be read off from table 4.2 for the fields we introduce in this model. The gauge invariant superpotential W_{torus} that reproduces the torus constraints has the form

$$\begin{aligned} W_{\text{torus}} =& \mathcal{C}_1 \sum_i \mathcal{Z}_{1i}^3 \mathcal{X}_{i11} + \mathcal{C}_2 \left(\prod_{i=1}^3 \mathcal{X}_{i11} \mathcal{Z}_{21}^3 + \sum_{j=2}^3 \mathcal{Z}_{2j}^3 \right) + \\ &+ \mathcal{C}_3 \left(\prod_{i=1}^3 \mathcal{X}_{i11} \mathcal{Z}_{31}^3 + \sum_{j=2}^3 \mathcal{Z}_{3j}^3 \right). \end{aligned} \tag{4.68}$$

We get the D-term constraints

$$|z_{a1}|^2 + |z_{a2}|^2 + |z_{a3}|^2 - 3|c_a|^2 = a_a, \tag{4.69}$$
$$|z_{1i}|^2 + |z_{21}|^2 + |z_{31}|^2 - 3|x_{i11}|^2 = b_{i11}, \tag{4.70}$$

with $a, i = 1, 2, 3$.

As before, the F-term constraints are obtained by taking to zero the derivatives of W_{torus} with respect to the scalar components of the chiral superfields involved. We get the three constraints $F_{x_{i11}}$

$$c_1 z_{1i}^3 + c_2 z_{21}^3 \prod_{j \neq i} x_{j11} + c_3 z_{31}^3 \prod_{j \neq i} x_{j11} = 0, \qquad i = 1, 2, 3. \tag{4.71}$$

Taking the derivative $\frac{\partial}{\partial z_{ai}} W_{\text{torus}}$, we get the nine $F_{z_{ai}}$-constraints

$$\begin{aligned} c_1 z_{1i}^2 x_{i11} = 0, \qquad c_2 z_{2j}^2 = 0, \qquad c_3 z_{3k}^2 = 0, \\ c_2 z_{21}^2 \prod_{i=1}^{3} x_{i11} = 0, \qquad c_3 z_{31}^2 \prod_{i=1}^{3} x_{i11} = 0. \end{aligned} \tag{4.72}$$

for $i = 1, 2, 3$ and $j, k = 2, 3$.

The F_{c_a}-terms read

$$x_{111} z_{11}^3 + x_{211} z_{12}^3 + x_{311} z_{13}^3 = 0, \tag{4.73}$$

$$\prod_i x_{i11} z_{a1}^3 + z_{a2}^3 + z_{a3}^3 = 0, \qquad a = 2, 3. \tag{4.74}$$

4.3.1 Divisors in the $x_{111}, x_{211}, x_{311}$ model

In accordance with equation (4.3), we get the symmetry actions for this model,

$$\theta_{111} = \theta, \qquad \theta_{211} = \theta \alpha_1, \qquad \theta_{311} = \theta \alpha_1^2. \tag{4.75}$$

We see that the actions θ and α_1 form a basis of all the $\mathbb{Z}_3$ actions we have around. Since the free action α_1 acts in only the first of the three two-tori, we obtain a factorized $A_2 \times A_2 \times A_2$ lattice. Once more, these symmetry actions have consequences on the definition of the divisors we consider: they all have to be invariant under θ and α_1. In general, the hypersurface constraints are the ones of the maximal fully resolvable model for divisors of the first two-torus and those of the minimal fully resolvable model for divisors that constrain coordinates of either one of the second two two-tori.

Ordinary divisors

In the first two-torus, the orbifold action θ is present as well as the 3-volution α_1. Furthermore, we can use the $\mathbb{C}^*$ symmetry of the weighted projective space we are in to form a complete basis of $\mathbb{Z}_3$-symmetries acting on the coordinates $z_{1i},\ i = 1, 2, 3$. As in the maximal fully resolvable model, we are free to change our basis of $\mathbb{Z}_3$-actions such that they act on the coordinates as

$$(z_{11}, z_{12}, z_{13}) \to (\zeta^i z_{11}, \zeta^j z_{12}, \zeta^k z_{13}), \tag{4.76}$$

where there is no further dependency between $i, j, k \in \{1, 2, 3\}$. In this form, it is obvious to see that the orbifold fixed points in the first two-torus correspond to $z_{1i} = 0$, which is what we use to define the three ordinary divisors of the first two-torus:

$$D_{1i} := \{z_{1i} = 0\} . \tag{4.77}$$

However, in the second two two-tori, the situation is different. We have no 3-volutions acting there, so for an orbifold fixed point it is indispensable that $z_{21} = z_{31} = 0$. Hence it is necessary for each of the three ordinary divisors of each of the last two two-tori to fulfill the condition $D_{an_a} := \{z_{a1} = 0\}$, $a = 2, 3$. As in the minimal fully resolvable model, this directly means that the condition $z_{a1} = 0$, $a = 2, 3$ does describe $\bigcup_{n=1}^{3} D_{an}$ rather than D_{a1} as it was in the maximal fully resolvable model. To pick one ordinary divisor out of the union of these three, we have to combine the condition z_{a1}, $a = 2$ or 3, with the F_{c_a}-term constraint (4.73) to get

$$\frac{z_{a2}}{z_{a3}} = -\zeta^{\tilde{n}}, \qquad \tilde{n} = 0, 1, 2. \tag{4.78}$$

The ordinary divisor D_{an} $a = 2, 3$ now is defined by

$$D_{an} := \{z_{a1} = 0 \mid \tilde{n} = n - 1\} . \tag{4.79}$$

Thus, in this model, it is important for the definition of divisors to keep in mind in which of the three two-tori we are.

Exceptional divisors

As already mentioned, the exceptional coordinates are never affected by the symmetries that act in the orbifold. Hence in the present model, the exceptional divisors E_{i11} are simply defined by

$$E_{i11} := \{x_{i11} = 0\} . \tag{4.80}$$

Inherited divisors

Because of the 3-volution α_1, the inherited divisor R_1 that fixes the coordinates in the first two-torus is defined by

$$R_1 := \bigcup_{k,l=0}^{2} \left\{ u_1 = \zeta^k \tilde{u}_1 + l \cdot \frac{\zeta - 1}{3} \right\} . \tag{4.81}$$

Hence it can be shown in a complete analogous way as in equations (4.14) - (4.20) that R_1 is a homogeneous cubic polynomial in the coordiantes z_{1i}, $i = 1, 2, 3$. The gauge invariant form is given by

$$a_{11}(\tilde{u}_1) x_{111} z_{11}^3 + a_{12}(\tilde{u}_1) x_{211} z_{12}^3 + a_{12}(\tilde{u}_1) x_{311} z_{13}^3 = 0, \tag{4.82}$$

with some coefficient functions a_{1i} depending on $\tilde{u}_1$.

On the other hand, since the divisors R_2 and R_3 need not to be 3-volution invariant, they are defined on the covering space as

$$R_a := \bigcup_{k=0}^{2} \left\{ u_a = \zeta^k \tilde{u}_a \right\}, \qquad a = 2, 3, \tag{4.83}$$

which, according to equations (4.39) - (4.45), translates in Weierstraß coordinates to

$$R_a := \{ a_{a2}(\tilde{u}_a) z_{a2} + a_{a3}(\tilde{u}_a) z_{a3} = 0 \}, \qquad a = 2, 3. \tag{4.84}$$

So, as for the ordinary divisors, in the first two-torus the inherited divisors are defined as in the maximal fully resolvable model, whereas in the other two two-tori they are defined as in the minimal fully resolvable model.

Equivalence relations

From equation (4.82) we can directly read off the equivalence relations

$$R_1 \sim 3D_{11} + E_{111} \sim 3D_{12} + E_{211} \sim 3D_{13} E_{311}. \tag{4.85}$$

For equivalence relations involving R_2 and R_3, we again have to combine their hypersurface constraints (4.84) with the torus defining equations (4.74) to get

$$\prod_i x_{i11} z_{a1}^3 + \left(1 - \frac{a_{a2}}{a_{a3}}\right) z_{a2}^3 = 0, \qquad a = 2, 3. \tag{4.86}$$

From this we get the equivalence relations

$$R_a \sim 3D_{ai} + \sum_i E_{i11}, \qquad a = 2, 3. \tag{4.87}$$

4.3.2 Intersection numbers

As in the maximal fully resolvable model, we confine ourselves to intersection numbers of the orbifold- and blow-up phase $0 < b << a$, where the orbifold singularities are resolved.

4.3.3 The orbifold phase

The orbifold phase is characterized by $b_{i11} < 0 < a_a$. As we have already seen, this forces the exceptional coordinates x_{i11} to be nonzero, hence there are no exceptional divisors $E_{i11} := \{x_{i11} = 0\}$ present in this phase. Thus, we can only compute intersection numbers involving ordinary and inherited divisors. Once more, the D- and F-term constraints force all c_a to vanish in the orbifold phase, so the only nontrivial F-term constraints are (4.73), (4.74). These are the constraints that we have to intersect with our divisor hypersurface equations. The number of distinct solutions to the emerging equation system is the intersection number of the corresponding divisors.

The torus-defining equation (4.73) of the first two-torus has an intersection set of 9 and 3 points with R_1 (4.82) and D_{1i} (4.77), respectively. Using the 3-volution α_1, these points are identified in groups of 3, so that the intersection number of R_1 with the first two-torus equation reduces to 3 and to 1 for the ordinary divisor D_{1i}.

In the other two two-tori, the intersection numbers of the divisors with the torus constraints (4.74) are also 3 and 1 for the inherited and ordinary divisors, respectively. However, the reason for that is another: since there is no 3-volution around, the inherited divisors are linear polynomials in z_{a2} and z_{a3}, as in (4.84). Hence the intersection set of this linear polynomial with the cubic polynomial of (4.74) directly leads to 3 solutions. Intersecting (4.74) with the necessary condition $z_{a1} = 0$ of the ordinary divisor D_{an}, we firstly get 3 solutions. But, as already mentioned, this corresponds to the intersection of the two-torus with $\bigcup_{n_a=0}^{2} D_{an_a}$, hence we have to divide by 3 and obtain 1 again.

To get the triple intersection numbers, we simply have to multiply these numbers of solutions and divide by 3 afterwards to consider the $\mathbb{Z}_3$-orbifold action. Hence we get again

$$\begin{aligned} R_1R_2R_3 &= \frac{3\cdot 3\cdot 3}{3} = 9, \qquad & R_1R_2D_{3n_3} &= \frac{3\cdot 3\cdot 1}{3} = 3, \\ R_1D_{2n_2}D_{3n_3} &= \frac{3\cdot 1\cdot 1}{3} = 1, \qquad & D_{1n_1}D_{2n_2}D_{3n_3} &= 1\cdot 1\cdot 1 = 1. \end{aligned} \tag{4.88}$$

The last intersection number, $D_{1n_1}D_{2n_2}D_{3n_3}$, has not to be divided by 3 for the orbifold action, since this intersection corresponds to an orbifold fixed point.

4.3.4 The blow-up phase

The blow-up phase is determined by $0 < b_{i11} << a_a$. The D- and F-term constraints (4.69) - (4.74) again force all c_a to be zero. Since $b_{i11} > 0$, exceptional divisors $E_{i11} := \{x_{i11} = 0\}$ are allowed. It is easy to show that an intersection of an inherited divisor R_a, $a = 1, 2, 3$ with an exceptional divisor E_{i11} is impossible because this would lead to a phase contradiction of the form $b_{i11} > a_a$.

We can compute the intersection numbers involving R_a, $a = 1, 2, 3$ and $D_{a'i}$ in exact the same way as we did in the orbifold phase and obtain the same results as in (4.88), with only one exception: the intersection $D_{1i}D_{2n_2}D_{3n_3}$ vanishes in the blow-up phase since this would contradict the phase condition $b_{i11} > 0$. Here we see again that the orbifold has been resolved, because $D_{1i}D_{2n_2}D_{3n_3}$ corresponds to an orbifold fixed point which is present in the orbifold phase, but vanishes in the blow-up phase.

By use of the linear equivalences (4.85) and (4.87), we get the intersection numbers of the exceptional divisors,

$$
\begin{aligned}
&E_{i11}^3 = 9, \qquad D_{1i}\sum_j E_{j11}^2 = D_{an_a}E_{i11}^2 = -3, \\
&D_{1i}D_{an_a}\sum_j E_{j11} = D_{2n_2}D_{3n_3}E_{i11} = 1, \qquad a = 2,3.
\end{aligned}
\tag{4.89}
$$

4.4 A non-factorized orbifold resolution model

The last model we discuss is the most subtle one: We consider a fully resolvable model with the exceptional gaugings $U(1)_{E_{111}}$, $U(1)_{E_{222}}$ and $U(1)_{E_{333}}$ and the corresponding exceptional coordinates $x_{111}, x_{222}, x_{333}$. According to (4.3), this model possesses the symmetry actions

$$
\theta_{111} = \theta, \qquad \theta_{222} = \theta\alpha_1\alpha_2\alpha_3, \qquad \theta_{333} = \theta\alpha_1^2\alpha_2^2\alpha_3^2. \tag{4.90}
$$

It is easy to see that the two independent actions θ and $\alpha_1\alpha_2\alpha_3$ form a basis of these symmetries. Since the 3-volution action $\alpha_1\alpha_2\alpha_3$ operates in all three two-tori simultaneously, we clearly do not have a factorized A_2^3 lattice anymore.

We try to go the naive way here, which means we define all our divisors such that they are invariant under the actions of (4.90). On the covering space, this means that the inherited divisors R_a are defined by union over the nine points

$$
R_a := \bigcup_{k,l=0}^{3} \left\{ u_a = \zeta^k \tilde{u}_a + l\frac{\zeta - 1}{3} \right\}. \tag{4.91}
$$

Hence on the cover, the intersection $R_1R_2R_3$ has $9\cdot 9\cdot 9 = 3^6$ distinct points. Using the orbifold- and 3-volution action of (4.90), we can identify these points in groups of 9, so we get the intersection number $R_1R_2R_3 = \frac{3^6}{3^2} = 81$. This is completely in contrast to all the other models we considered so far, where we always had $R_1R_2R_3 = 9$.

This conflict can be solved as follows [13]: first, we have to determine a refined torus lattice that contains the shift $\alpha_1\alpha_2\alpha_3$ as a lattice vector. Then we redefine the fundamental domain of our orbifold to be the parallelepiped spanned by the lattice vectors of the refined lattice. Finally, we count how many branches each of our divisors has under the actions (4.90) within this new fundamental domain. We multiply them and divide out the size of the symmetry group defined by (4.90) as we identify points that differ by combinations of θ and $\alpha_1\alpha_2\alpha_3$.

4.4.1 The torus lattice of the $x_{111}, x_{222}, x_{333}$ model

The A_2^3 lattice of the resolution models of $T^6/\mathbb{Z}_3$ we treated so far is given by [7]

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \\ 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, e_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix},$$

since we have chosen our two-tori to be spanned by 1 and the fixed complex structure $\zeta = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. To get the refined lattice that fits to our model, the vector of the 3-volution action $\alpha_1\alpha_2\alpha_3$ has to be included to the torus lattice. Since $\alpha_1\alpha_2\alpha_3$ acts as

$$u_a \sim u_a - \frac{1}{2} + \frac{\sqrt{3}}{6}i, \qquad a = 1, 2, 3, \tag{4.92}$$

the new lattice vector is

$$e_{\alpha_1\alpha_2\alpha_3} = \frac{1}{3}\left(e_2 - e_1 + e_4 - e_3 + e_6 - e_5\right). \tag{4.93}$$

So we get the new lattice $\hat{e}_i$ [7]

$$\begin{aligned} &\hat{e}_1 = e_3 + e_4, \qquad \hat{e}_2 = -e_4, \qquad \hat{e}_3 = \frac{1}{3}\left(e_2 - e_1 + e_4 - e_3 + e_6 - e_5\right) \\ &\hat{e}_4 = -e_6, \qquad \hat{e}_5 = e_5 + e_6, \qquad \hat{e}_6 = -e_2. \end{aligned} \tag{4.94}$$

Note that each of the old lattice vectors e_i can be written as an integer relation of the new lattice vectors $\hat{e}_i$.

A well-defined orbifold lattice has to be invariant under the orbifold action. This means that the lattice vectors of the refined lattice have to satisfy the relations [7]

$$\begin{aligned} e_{2a-1} &\xrightarrow{\theta} e_{2a}, \\ e_{2a} &\xrightarrow{\theta} -e_{2a} - e_{2a-1}, \qquad a = 1, 2, 3. \end{aligned} \tag{4.95}$$

Demanding this to our lattice basis $\hat{e}_i$, we end up with the final refined lattice vectors $\tilde{e}_i$ for this model [7]:

$$\begin{aligned} &\tilde{e}_1 = e_3, \qquad \tilde{e}_2 = e_4, \qquad \tilde{e}_3 = \frac{1}{3}\left(-e_1 + e_2 - e_3 + e_4 - e_5 + e_6\right), \\ &\tilde{e}_4 = -\frac{1}{3}\left(e_1 + 2e_2 + e_3 + 2e_4 + e_5 + 2e_6\right), \qquad \tilde{e}_5 = e_5, \qquad \tilde{e}_6 = e_6. \end{aligned} \tag{4.96}$$

By computation of the Cartan matrix $A_{mn} = 2\frac{\tilde{e}_m \cdot \tilde{e}_n}{\tilde{e}_m \cdot \tilde{e}_m}$, we can classify this lattice to be an E_6 Lie-lattice.

4.4.2 The new fundamental domain

We choose the parallelepiped spanned by the vectors $\tilde{e}_i$, $i = 1, ..., 6$ to be the fundamental domain of the orbifold we are investigating. This fundamental domain can be parametrized by

$$T^6_{E_6} := \{ \sum_{i=3}^{6} x_i e_i + \frac{1}{3} y_1(-e_1 + e_2 - e_3 + e_4 - e_5 + e_6) \\ - \frac{1}{3} y_2(e_1 + 2e_2 + e_3 + 2e_4 + e_5 + 2e_6) \mid 0 \leq x_i, y_1, y_2 < 1\}. \tag{4.97}$$

An inherited divisor R_a fixes the cover coordinate u_a to some value $\tilde{u}_a$, hence we have to give a mapping from the coordinates x_i, y_1, y_2, $i = 3, ..., 6$ back to the cover coordinates u_a in order to parametrize the fundamental domain $T^6_{E_6}$ in the coordinates u_a. This mapping is given by

$$\begin{aligned} u_1 &= -\frac{1}{2} y_1 + \frac{\sqrt{3}}{6} i(y_1 - 2y_2), \\ u_2 &= x_3 - \frac{1}{2} x_4 - \frac{1}{2} y_1 + \frac{\sqrt{3}}{2} i(x_4 + \frac{1}{3} y_1 - \frac{2}{3} y_2), \\ u_3 &= x_5 - \frac{1}{2} x_6 - \frac{1}{2} y_1 + \frac{\sqrt{3}}{2} i(x_6 + \frac{1}{3} y_1 - \frac{2}{3} y_2). \end{aligned} \tag{4.98}$$

Plugging in the extremal values for x_i and y_1, y_2, we can read off the ranges of the coordinates u_a:

$$\begin{aligned} -\frac{1}{2} &< Re(u_1) \leq 0, & -\frac{\sqrt{3}}{3} &< Im(u_1) < \frac{\sqrt{3}}{6}, \\ -1 &< Re(u_2) < 1, & -\frac{\sqrt{3}}{3} &< Im(u_2) < \frac{2\sqrt{3}}{3}, \\ -1 &< Re(u_3) < 1, & -\frac{\sqrt{3}}{3} &< Im(u_3) < \frac{2\sqrt{3}}{3}. \end{aligned} \tag{4.99}$$

It is important to note that the fundamental domain $T^6_{E_6}$ is *not* given by the cuboid spanned by the ranges of the complex coordinates u_a, because the coordinates y_1, y_2 appear in each coordinate u_a. Nevertheless, an inherited divisor R_a can fix the coordinate u_a to any value in between its coordinate range (the orbifold fixed points excluded). However, it might happen that the intersection of the inherited divisors does not lie in the fundamental domain $T^6_{E_6}$ anymore. These intersections can be mapped back to the fundamental domain using the torus vectors. As we see in figure 4.1, such intersection points become identified with already existing intersection points within the fundamental domain.

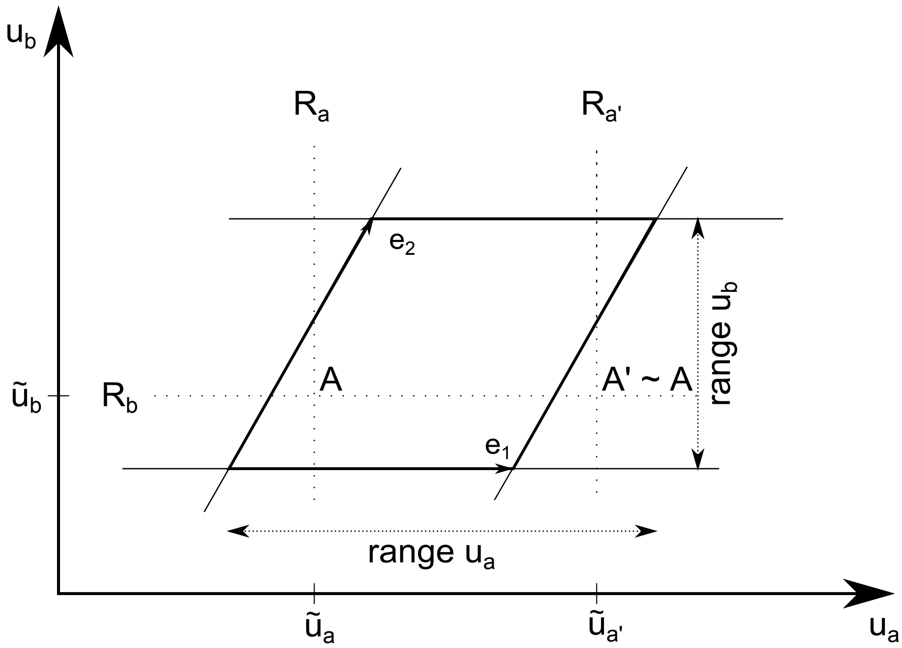

Figure 4.1: Two (real) inherited divisors R_a, R_b in a (real) two-dimensional torus T^2. We see that the divisor R_a splits in two parts, which is why we get two intersection points A, A' that differ by the torus lattice vector e_1. Hence these two intersection points A, A' are identified, $A \sim A'$.

4.4.3 The branch counting

First, we look how many 3-volution branches each inherited divisor R_a has within its coordinate range given in (4.99). Afterwards, we see which of these branches can be identified using the torus lattice vectors.

The 3-volution acts as

$$\alpha_1\alpha_2\alpha_3 : \qquad u_a \to u_a - \frac{1}{2} + \frac{\sqrt{3}}{6}i, \qquad a = 1, 2, 3, \tag{4.100}$$

so we directly see that R_1 can only have one single 3-volution branch: the range of the real part of u_1 only has the length $\frac{1}{2}$, so there can not be any further 3-volution image of R_1 within the fundamental domain we have chosen, because the action $\alpha_1\alpha_2\alpha_3$ already shifts the real part by $-\frac{1}{2}$. To count the 3-volution branches in the second and third two-tors, we look at figure 4.2. We see that in these two-tori, each inherited divisor has three 3-volution branches. This has its reason in the fact that the action of $\alpha_1\alpha_2\alpha_3$ in the 2. (3.) two-torus satisfies the equation

$$3(\alpha_1\alpha_2\alpha_3)_a = \begin{pmatrix} -\frac{3}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_{2a} - e_{2a-1}, \qquad a = 2, 3. \tag{4.101}$$

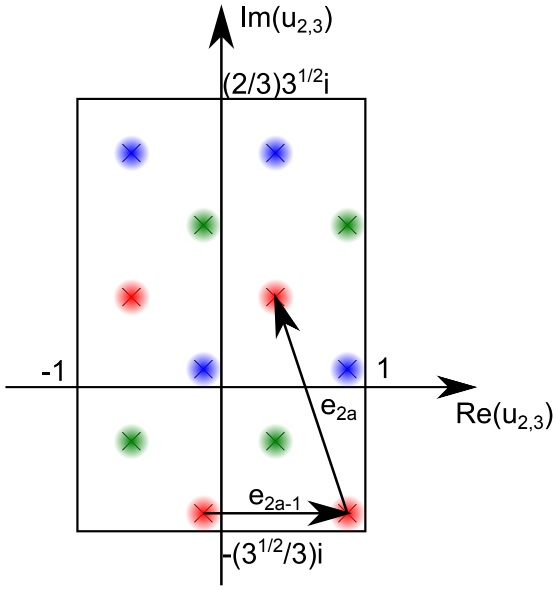

Figure 4.2: One inherited divisor $R_{2,3}$ in the coordinate range of the second/third two-torus and all its 3-volution branches. Points that are coloured the same can be identified using the torus lattice vectors. As we see, we have 3 distinct 3-volution branches (one for each colour).

Hence every third 3-volution can be expressed by an integer relation of the lattice vectors e_{2a}, e_{2a-1}, so there are three independent 3-volution branches.

Next we investigate how many orbifold action θ branches there are of an inherited divisor within the fundamental domain (4.97) we have chosen. We see in (4.95) that the orbifold action θ always maps points from the inside to the outside of the fundamental domain. But, we can use the lattice vectors $\tilde{e}_1, \tilde{e}_2, \tilde{e}_5$ and $\tilde{e}_6$ to shift them back to the fundamental domain we defined. The lattice vectors $\tilde{e}_3, \tilde{e}_4$ can not be used to shift points back to the fundamental domain since they would induce a coordinate shift in the other two two-tori, which is not allowed. Hence, we get a single orbifold branch in the first two-torus, whereas we have three of them in the second two two-tori.

We can multiply the number of branches to get the intersection number $1 \cdot 3 \cdot 3 \cdot 1 \cdot 3 \cdot 3 = 3^4$ on the cover. Dividing out the orbifold- and 3-volution action θ and $\alpha_1\alpha_2\alpha_3$, we get

$$R_1 R_2 R_3 = \frac{1 \cdot 3 \cdot 3 \cdot 1 \cdot 3 \cdot 3}{3 \cdot 3} = 9, \tag{4.102}$$

as we already had in the other resolution models on a factorized A_2^3 lattice.

Chapter 5

Conclusion

5.1 Summary

We considered the six target space extra dimensions of superstring theory to be compactified on an orbifold or rather a smooth Calabi-Yau resolution space thereof. Such a resolution space can either be obtained using toric geometry (described in chapter 3, furthermore in [9], [12], [16]) or by means of a Gauged Linear Sigma Model (GLSM) [7]. In particular, we focussed on GLSM resolutions of the $T^6/\mathbb{Z}_3$ orbifold. We considered the underlying six-torus to be factorizable as $T^6 = T^2 \times T^2 \times T^2$. As we knew from [7], such six-tori T^6 can be described as the intersection set of three elliptic curves in a toric variety X ($X = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$), one curve for each two-torus. In a GLSM setting, the hypersurface equations of these elliptic curves can be recovered as the roots of the derivatives of a superpotential W_{torus} with respect to the scalar components of some superfields $\mathcal{C}_a$ (the so-called F_{c_a}-term constraints).

By introduction of exceptional coordinates x at appropriate places and a gauge fixing of their phases (if they are nonzero), we got back the symmetry actions of the orbifold of the resolution model we chose. We saw that different resolution models in general lead to different symmetries acting in the orbifold. In detail, there are free symmetry actions, called 3-volutions α_a, that are not present in every resolution model.

This had important conseuqences on how one has to define the equations of the complex codimension 1 hypersurfaces in $T^6/\mathbb{Z}_3$ called divisors, because each divisor has to be invariant under all symmetry actions the orbifold possesses. As we computed the intersection numbers of three such divisors in each resolution model anew, we observed that they do not depend on the resolution model chosen.

In contrast, using the so-called minimal fully resolvable model, we saw that the intersection numbers may change varying the FI-parameters of the resolution model from one specific interval to another.

Furthermore, we saw in section 4.4 that there are resolution models that lead to non-factorized torus lattices, i.e. we do not have a six-torus that is writeable as $T^6 = T^2 \times T^2 \times T^2$ anymore. In such resolution models, the usual GLSM fashion we used to compute intersection numbers breaks down. This seems to be in correlation with the fact that the Weierstraß mapping described in section 2.1.1 can only be generalized to six-tori that can be factorized as $T^6 = T^2 \times T^2 \times T^2$, which is not the case in 4.4. Anyway, using another reasoning, it was possible to compute an intersection number in a rather extensive way without using any GLSM methods.

5.2 Outlook

We were only able to compute intersection numbers in a GLSM way using resolution models that lead to factorized six-tori. A consistent way to compute intersection numbers even for non-factorized orbifold resolution models has still to be worked out. The key point to this may eventually be the description of the six-torus as elliptic curves in the space $X = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, which is only valid for factorized six-tori. Perhaps one could find a similar description even for non-factorized six-tori which would allow to compute intersection numbers in the same algebraic way as for resolution models with factorized torus lattices.

Furthermore, it is an open task to give a strictly mathematical derivation of the equivalence relations between divisors in toroidal orbifolds. It would be very helpful to have a reliable standard procedure at hand that gives such equivalence relations for each orbifold in each resolution model.

Clearly, it is possible to extend the discussed GLSM methods to compute intersection numbers to other orbifolds. In a minimal fully resolvable model, it would be realizable to give intersection numbers in resolution phases that are beyond the orbifold and blow-up regime, as it was done in section 4.2.2 for $T^6/\mathbb{Z}_3$.

Appendix A

Intersection numbers - summary

We give a compact summary of the intersection numbers found in various phases in the minimal fully resolvable model, section 4.2.2. Unlisted intersections vanish.

$a, b < 0$: The non-geometric regime

There are no nontrivial divisor intersections.

$b < 0 < a$: The orbifold phase

$$\begin{aligned} R_1R_2R_3 &= 9, & R_1R_2D_{3n_3} &= 3, \\ R_1D_{2n_2}D_{3n_3} &= 1, & D_{1n_1}D_{2n_2}D_{3n_3} &= 1. \end{aligned} \tag{A.1}$$

$0 < b < a$: Blow-up phase I

$$\begin{aligned} R_1R_2R_3 &= 9, & R_1R_2D_{3n_3} &= 3, \\ R_1D_{2n_2}D_{3n_3} &= 1, & D_{1n_1}D_{2n_2}D_{3n_3} &= \text{ nonex.}, \\ D_{1n_1}D_{2n_2}E &= 1, & D_{an_a}E^2 &= -3, \qquad E^3 = 9. \end{aligned} \tag{A.2}$$

$0 < a < b < 2a$: Blow-up phase II

$$\begin{aligned} R_1R_2R_3 &= 9, & R_1R_2D_{3n_3} &= 3, \\ R_1R_2E &= 0, & R_1D_{2n_2}E &= 3, \\ D_{an_a}E^2 &= -3, & R_1E^2 &= 0, \qquad E^3 = 9. \end{aligned} \tag{A.3}$$

$0 < 2a < b < 3a$: **The critical blow-up phase,** $c_1, c_2 \neq 0$, $c_3 = 0$

$$R_3 = 1, \qquad D_{3n_3} = 1. \tag{A.4}$$

$0 < 2a < b < 3a$: **The critical blow-up phase,** $c_a = 0\ \forall a = 1, 2, 3.$

$$\begin{aligned} R_1R_2R_3 = 9, &\qquad R_1R_2D_{3n_3} = 3, \\ R_1R_2E = 0, &\qquad R_1D_{2n_2}E = 3, \\ D_{an_a}E^2 = -3, &\qquad R_1E^2 = 0, \qquad E^3 = 9. \end{aligned} \tag{A.5}$$

$0 < 3a < b$: **The over-blow-up phase,** $c_a \neq 0\ \forall a = 1, 2, 3.$

There are no nontrivial divisor intersections.

$0 < 3a < b$: **The over-blow-up phase,** $c_1, c_2 \neq 0$, $c_3 = 0$

$$R_3 = 1, \qquad D_{3n_3} = 1. \tag{A.6}$$

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Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

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